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Astronomia mechanica

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XI.

Astronomia mechanica.

Caput I.

De viribus, quibus corpora coelestia sollicitantur.

1. **Definitio 1.** *Astronomia mechanica* est scientia motus corporum coelestium ex viribus, quibus sollicitantur, determinandi.

2. **Coroll. 1.** Cognitis viribus, quibus corpora coelestia sollicitantur, eorum motus per principia mechanica determinatur, ex quo haec scientia nomen Astronomiae mechanicae est adepta.

3. **Coroll. 2.** Ante omnia ergo vires nosse oportet, quibus corpora coelestia sollicitantur, quarum cognitio cum ex suis causis cognosci nequeat, earum indolem ex ipsis phaenomenis scrutari convenit.

4. **Coroll. 3.** Si enim corpora coelestia a nullis viribus sollicitarentur, singula vel quiescerent, vel uniformiter in directum progrederentur, secundum prima Mechanicae principia.

5. **Scholion 1.** Quatenus scilicet corpora coelestia non uniformiter in directum incedunt, sed vel celeritate variata, vel secundum lineas curvas progrediuntur, eatenus vires adsint necesse est, quibus eorum motus afficiatur. Atque hoc non solum de motu absoluto valet, sed etiam de respectivo, cum semper vires concipi possint, quibus motus cujusquam corporis, qualis ex alio apparet, producatur. In Astronomia autem non tam motus corporum coelestium absolutos spectamus, quam eos, quibus spectatori in quopiam loco constituto moveri videntur, etiamsi forte hic ipse locus quomodocunque circumferatur. Hic ergo locus spectari solet tanquam absolute quiescens, ejusque respectu corporum coelestium motus ita considerari debent, ut vires, quae talem motum producere valeant, definiantur. Ita vires assignari possent, quae planetis eos motus irregulares inducerent, quibus ex terra visi incedere observantur, verum hoc modo illae vires nimis prodirent complicatae, quam ut earum existentiam in mundo agnoscere liceret. Quam ob causam talem investigationem respectu plurium locorum, in quibus spectator concipiatur, suscipi conveniet, et pro quo in viribus inventis maxima simplicitas inesse deprehenditur, eas demum tanquam in mundo revera existentes

admittere fas erit. Dummodo enim vires invenerimus, quae pro dato spectatoris loco coelestibus producendis fuerint pares, eas tanquam revera existentes considerare possumus, forte ob motum spectatoris longe aliae vires in mundo existerent, quandoquidem hic nobis est propositum motus apparentes explicare.

6. **Scholion 2.** Ad hanc ergo virium investigationem instituendam nosse oportet motus porum coelestium, qui quo accuratius fuerint perspecti, eo certius illarum virium indolem cog licebit. Quare si quilibet motus peculiarem virium legem postularet, ita ut inde pro reliquis concludere liceret, hinc etiam ipsa motuum cognitio nihil lucri adipisceretur; sin autem eve vires pro omnibus motibus inventae ad communem quandam regulam referantur, hinc sine plurimum lucis consequemur; cum inde etiam eorum motuum, quorum ratio per observationes satis fuerit explorata, explicatio peti queat. Hocque ergo casu ex Astronomia mechanica incrementa in Astronomiam practicam transferrentur. Atque istud institutum summus quondam Isaacus Newtonus felicissimo successu absolverit, dum ex collatione phaenomenorum cum pr mechanicis elegantissimam aeque ac simplicissimam legem detexit, quae omnes vires coelestes com teretur. Haec itaque virium lex fundamentum universae Astronomiae mechanicae constituet omnium motuum coelestium ratio est repetenda, quod felicissimum inventum, cum phaenomen tissimum innitatur, instar hypothesis hic proponam et ad usum accomodabo.

7. **Hypothesis 1.** *Corpora coelestia perinde inter se commoventur, ac si singula se attraherent, viribus reciproce quadratorum distantiis proportionalibus.*

8. **Explicatio.** De tellure res est manifesta, cum omnia corpora circa terram existenti sum urgeantur vi, quae gravitas appellatur, et sine dubio ad multo majores distantias extend quam experimenta capere licet. Luna enim ab eadem vi sollicitari deprehenditur, quae autem minor est quam prope terrae superficiem, hocque fere in ratione reciproca duplicata dist Scilicet cum luna circiter sexagies magis sit remota a centro terrae quam corpora in ejus sup sita, vis, qua luna terram versus urgetur, sexagies minor est aestimanda quam vis gravit superficie, quemadmodum ex motu lunae concludere licet. Gravitas quidem in superficie terr effectus mixtus ex vero corporum nisu deorsum directo et vi centrifuga e motu terrae diurno unde potissimum evenit, ut gravitas neque ubique praecise ad centrum terrae dirigatur, neque dem sit magnitudinis. Seposita autem vi centrifuga, quippe cujus corpora longius a terra sunt expertia, nisus ad ejus centrum directus satis exacte distantiarum ab eo quadratis r proportionalis deprehenditur. Concesso igitur hoc, quod omnia corpora, quantumvis a terra remota, ad ejus centrum quasi trahantur hujusmodi vi, similes vires Newtonus singulis cor mundanis tribuit, ita ut eorum quodque reliqua corpora ad se attrahat viribus in ratione dupl distantiarum decrescentibus. Atque in hac lege contineri sunt censendae omnes vires, quibus porum coelestium motus reguntur.

9. **Coroll. 1.** Cum istae vires attractrices corporum mundanorum in ratione duplicata tiarum decrescant, in maximis distantiis tam parvae evadunt, ut pro evanescentibus haberi q

10. **Coroll. 2.** Hinc cum stellae fixae ad tam enormes distantias a nobis et toto sys solari sint remotae, vires, quibus sol et planetae ad stellas fixas attrahuntur, pro nihilo sunt hab

11. **Coroll. 3.** Planetarum igitur et cometarum motus a nullis aliis viribus perturbari sunt nisi quibus vel ad solem vel ad reliquos planetas gravitatione mutua sollicitantur.

12. **Scholion 1.** Haec virium mundanarum lex per phaenomena stabilita etsi in magnis intervallis veritati apprime consentanea deprehenditur, tamen in minoribus distantiis, praecipue ubi corpus attractum intra superficiem corporis attrahentis est situm, a veritate vehementer recedit. Evanescente enim distantia corporis attracti a centro corporis attrahentis, secundum hanc legem vis attrahens infinita prodiret, id quod sine dubio rerum naturae adversaretur. Quin potius statuere debemus, si in terrae visceribus experimenta capere liceret, quo propius ad ejus centrum pertingeremus, eo minorem esse futuram vim gravitatis, cum in ipso centro plane in nihilum abire necesse sit ob defectum rationis, cur potius in hanc quam aliam plagam dirigeretur. Ex quo etiam Newtonus statuit, a terrae superficie intus ad ejus centrum penetrando, vim gravitatis iterum decrescere atque ipsis a centro distantiis esse proportionalem, quem saltum ex principiis gravitationis universalis egregie explicavit, ita ut lex illa ad corpora extra se posita nullum detrimentum patiatur. Cum igitur in Astronomia corpora tantum longis intervallis a se invicem dissita considerentur, sine ulla haesitatione phaenomena sequentes agnoscere debemus vires, quibus ea se mutuo attrahunt, rationem reciprocam duplicatam distantiarum sequi, atque esse directas ad cujusque corporis attrahentis centrum, siquidem figuram habeant sphaericam, vel potius centrum inertiae, si ab hac figura abludant, quod quidem exiguum discrimen in tantis distantiis pro nihilo est habendum. Quamvis enim in superficie terrae gravitas non ubique ad ejus centrum dirigatur, tamen in grandibus ab ea distantiis, etiamsi gravitatis directio aliquantillum centrum praetergrediatur, haec tantilla aberratio in motuum determinatione nullius plane est momenti.

13. **Scholion 2.** Quando autem statuimus singula corpora coelestia perinde ac terram ejusmodi proprietate esse praedita, qua corpora extra se posita attrahant vi quadrato distantiae reciproce proportionali, haec virium ratio de eadem vel aequalibus massis corporis attracti est interpretanda. Posita scilicet massa corporis attracti $= M$, ejusque distantia a centro corporis attrahentis $= x$; si vis qua eo impellitur fuerit V, eadem massa M in alia distantia $= y$ a centro ejusdem corporis attrahentis remota eo impelletur vi U, ut sit

$$V : U = \frac{1}{xx} : \frac{1}{yy} \quad \text{seu} \quad U = \frac{xx}{yy} V.$$

Unde si pro una quadam distantia innotescat vis attractrix, pro alia quacunque distantia facile definietur, siquidem ambo corpora tam attrahens quam attractum maneant eadem. Sin autem massa corporis attracti fuerit major vel minor, vis attractrix praeterea in eadem ratione erit augenda vel minuenda, ut mox declarabimus. Tum vero etiam vis attrahens diversorum corporum coelestium plurimum discrepat, ita ut etiam in pari distantia diversas vires exerant. Scilicet cum vis attrahens terrae in superficie, seu distantia ab ejus centro, radio ejus aequali, sit ipsa gravitas, pro aliis corporibus coelestibus distantia, ad quam vis eorum attractrix gravitati est aequalis, modo major modo minor esse potest, quam radius terrae; unde dicimus alia corpora coelestia majori, alia minori facultate attrahendi pollere, etiamsi pro unoquoque vis attrahens sequatur rationem reciprocam duplicatam distantiarum.

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14. **Hypothesis 2.** *Vires, quibus diversa corpora ab eodem corpore coelesti in ead... stantia attrahuntur, sunt ut eorum massae, ac si distantiae fuerint diversae, rationem seq... compositam ex ratione massarum et reciproce duplicata distantiarum.*

15. **Coroll. 1.** Vis ergo attractrix corporum coelestium perinde ac gravitas terrestris sec... quantitatem materiae in corpora agit, ita ut nisus cujusque corporis sit ejus massae propor... siquidem distantia fuerit eadem.

16. **Coroll. 2.** Hae igitur vires coelestes corpora quasi penetrant, in eaque, quatenus ... sunt praedita, agunt, ita ut quo major fuerit massa cujuspiam corporis, ideo fortius a qu... corpore coelesti attrahatur.

17. **Coroll. 3.** Hinc vis, qua corpus quodvis ad aliquod corpus coeleste attrahitur... aggregatum omnium virium elementarium, quibus singula corporis elementa pro ratione massae... inertiae sollicitantur.

18. **Scholion.** De corporibus circa terram sitis per experimenta est evictum eorum gr... tatem seu pondus ipsorum massae esse proportionale, cum singulae particulae seorsim gravit... ratione massae. Quare cum vis attractrix corporum coelestium pari ratione sit comparata atque... gravitatis telluris, nullum etiam est dubium, quin eorum vires parem sequantur legem atque co... attracta pro ratione materiae seu massae afficiunt. Unde si singulae corporis particulae a corp... coelesti aeque distent, quod evenire censendum est, si magnitudo corporis attracti prae ejus dist... a centro corporis attrahentis tam sit exigua ut pro nihilo reputari queat, singulae vires elem... erunt aequales sub directionibus inter se parallelis: hincque vis tota earum summae aequalis, ej... directio per centrum inertiae corporis attracti transire est censenda. Sin autem corporis att... magnitudo ad distantiam a centro corporis attrahentis notabilem teneat rationem, ut vires elem... tares, quibus singula corporis elementa attrahuntur, neque pro aequalibus, ob inaequales element... distantias, neque directiones pro parallelis inter se haberi queant, vis tota inde demum per calcul... est colligenda; hincque evenire potest, ut vis tota neque massae corporis attracti sit proportion... neque per ejus centrum inertiae transeat. Ex quo proprie loquendo ambae allatae hypotheses tant... ad corpuscula quasi infinite parva, quae ad corpora coelestia attrahantur, sunt restringendae, ... vires tantum elementares praefatas leges sequantur. Unde deinceps vires totae, quibus corpor... jora attrahuntur, per regulas staticas demum colligi debeant.

19. **Hypothesis 3.** *Corpora coelestia in aequalibus distantiis eo majores exerunt vires att... trices, quo majores fuerint ipsorum massae, atque in inaequalibus distantiis vires attractrices corp... coelestium sunt in ratione composita massarum et reciproca duplicata distantiarum.*

20. **Coroll. 1.** Hinc si massa corporis coelestis fuerit $=A$, in distantia x ab ejus cen... erit vis attractrix ut $\frac{A}{xx}$, quae propterea est directe ut massa corporis attrahentis A, et recipr... quadratum distantiae ab ejus centro.

21. **Coroll. 2.** Haec autem lex tantum valet, si corporis attracti massa fuerit eadem; ... quin enim cum illa ratione composita insuper ratio massae corporis attracti conjungi debet, secun... dum hypothesin praecedentem.

22. **Coroll. 3.** Ita si massa corporis coelestis sit $=A$, corporis autem attracti massa $=M$, quaeque ab illius centro distantia $=x$, erit vis, qua hoc ad illud attrahitur ut $\frac{AM}{xx}$. Quia autem in motus determinatione vis haec semper per massam corporis attracti M dividi debet, in calculum primum ingreditur formula $\frac{A}{xx}$, massa M inde iterum egrediente.

23. **Scholion 1.** Ista hypothesis non aeque certo ex phaenomenis colligitur ac praecedentes, nisi enim inde veram virium quantitatem concludere licet, tamen nulla constat ratio massas corporum coelestium dignoscendi. Eorum quidem magnitudinem Astronomi sollicite definire conantur, verum hinc de massa, quoniam pro volumine vehementer discrepare potest, nihil statuere licet. Haec massae autem ratione omnino carere possemus, si pro quovis corpore coelesti eam distantiam determinemus, in qua vis ejus attractrix aequalis esset futura vi gravitatis in superficie terrae. Ita si pro quopiam corpore coelesti haec distantia fuerit $=f$, pro alia quacunque distantia x ejus vis attractrix erit $=\frac{ff}{xx}$ unitate gravitatem exprimente: Sicque sufficit pro quolibet corpore coelesti distantiam hanc f ipsi convenientem nosse, parumque interest quomodo haec distantia ad massam corporis attrahentis se sit habitura. Newtonus autem statuit quadratum istius distantiae semper esse massae corporis attrahentis proportionale; ita ut loco ff massa A substitui queat, — quae propositio, si de ejus veritate convicti essemus, non solum pro pulcherrima esset habenda, sed etiam plurimum lucis ad naturae mysteria scrutanda esset allatura. Phaenomena autem huic propositioni jam insignem probabilitatis gradum conciliant, quoniam quo corpora coelestia fuerint majora vel minora, etiam quadratum distantiae f illis respondentis majus minusve deprehenditur: ratione autem ea propositio multo magis confirmari potest. Primum enim cum gravitatio in corporibus coelestibus sit reciproca, ex principio aequalitatis inter actionem et reactionem duo corpora se mutuo attrahentia paribus viribus in se invicem niti debent. Hinc si eorum massae sint A et B, et distantiae, ad quas eorum vires attractrices gravitati aequantur, sint a et b, distantia inter centra corporum existente $=x$, erit per praecedentem hypothesin vis, qua corpus B ad A urgetur $=\frac{aaB}{xx}$, vis autem, qua corpus A vicissim ad B urgetur $=\frac{bbA}{xx}$, quae duae vires si censeantur aequales, fit $aa:bb=A:B$, prorsus ut vult Newtonus. Quodsi haec propositio admittatur, inde etiam haec eximia proprietas consequitur, quod plurium corporum se mutuo hac lege attrahentium commune centrum inertiae perpetuo vel quiescat, vel uniformiter in directum proferatur; — quae cum tanquam constans naturae lex assumenda videatur, inde vicissim veritas nostrae hypotheseos evincitur.

24. **Scholion 2.** Quoniam igitur vis attractrix massae corporis attrahentis est proportionalis, ea tantum inde oriri est judicanda, ut singula corporis elementa seorsim vires attractrices exerant, ex quarum collectione demum vis tota attrahens nascatur. Lex ergo praescripta tantum ad vires elementares, quibus singula corporis elementa ad se attrahuntur, proprie pertineri est censenda, neque tribui ad corpora finita extendi patitur, nisi haec corpora tantopere a se invicem fuerint remota, ut eorum magnitudo prae distantia quasi evanescat. Eatenus ergo tantum haec lex attrahendi in corporibus mundanis deprehenditur, quatenus ea tam vastis intervallis a se invicem sunt sejuncta, quae si sibi multo essent propiora, nullum est dubium quin vires eorum attractrices ab hac lege sint

aberraturae, nisi forte eorum figura perfecte fuerit sphaerica. Quare si hoc modo praec[illegible] hypothesin restringamus, uti ratio suadet, universum Astronomiae mechanicae fundamentum [illegible] hypothesi continebitur.

25. **Hypothesis 4.** *Omnia materiae elementa, ex quibus corpora mundana sunt conf[illegible] pollent attrahendi, quae cujusque massae directe, inverso autem quadrato distantiae est propor[illegible] cum qua ratione insuper massa corpusculi attracti est conjungenda.*

26. **Coroll. 1.** Positis ergo massis duorum elementorum materiae μ et ν, eorumque di[illegible] $= z$, erit vis, qua alterum ab altero attrahitur ut $\frac{\mu\nu}{zz}$, atque ex viribus elementaribus, quibus [illegible] duorum corporum finitorum se mutuo attrahunt, vires ambo corpora tota sollicitantes, debent [illegible]

27. **Coroll. 2.** Cum haec vis elementaris sit ut $\frac{\mu\nu}{zz}$, factor quidam constans C adjung[illegible] ut formula $\frac{C\mu\nu}{zz}$ ipsam vim exprimat; atque haec quantitas C perpetuo erit eadem ubicunque [illegible] ista duo elementa materiae fuerint posita.

28. **Coroll. 3.** Corpora ergo coelestia eatenus tantum se mutuo attrahunt, quatenus con[illegible] materia, cujus omnia elementa tali vi se attrahendi sunt praedita. Ex quo patet si corpor[illegible] fuerint admodum a se invicem remota, simulque figuram habeant irregularem, fieri posse, ut vi[illegible] e viribus elementaribus resultans multum a simplicitate formulae exhibitae discrepet.

29. **Scholion 1.** Cum hujusmodi vis attrahendi omnibus corporibus coelestibus conven[illegible] ideoque omnibus elementis corporum, ubicunque reperiantur, tribui debeat, ea tanquam pro[illegible] universalis materiae spectari potest, cujus existentiam realem ex phaenomenis cum ratione con[illegible] evictam agnoscere debemus. Certum itaque est omnia materiae elementa tali vi attrahendi, qual[illegible] descripsimus, esse praedita, neque de hoc ullo modo dubitare licet. Utrum autem haec vis [illegible] materiae ex sua natura competat perinde atque inertia et impenetrabilitas? an vero a causa qua[illegible] externa producatur? multum inter Philosophos dubitatur. Qui priorem sententiam propugnant [illegible] mium firmamentum inveniunt in universalitate istius indolis attractricis, quae cum in omnibus co[illegible] poribus inesse, atque adeo eorum massae proportionalis deprehendatur, eam pariter atque ine[illegible] materiae essentialem esse autumant, ita ut ejus causa extrinsecus frustra quaeratur; quin etiam [illegible] rere solent, an non Creator per omnipotentiam corporibus talem proprietatem infundere potue[illegible] quod negare ipsis adeo impium videtur. Deinde in hoc non parum subsidii sibi situm esse arbi[illegible] tur, quod nemo adhuc istius phaenomeni latissime patentis causam externam dilucide docere vol[illegible] Qui autem contrariae sententiae sunt addicti, haec argumenta gravibus rationibus infirmare co[illegible] maluntque credere dari hujus phaenomeni causam externam, etiamsi eam nobis nullo modo persp[illegible] liceat, quam concedere ejus rationem in ipsa materiae indole esse positam. Ad nostrum [illegible] institutum parum refert, utrum causa externa existat, nec ne? sufficit enim nosse in omnibus co[illegible] poribus mundi talem vim attrahendi revera inesse, cum nobis id tantum sit propositum, ut quo[illegible] motus corporum coelestium ab his viribus afficiantur, investigemus.

30. **Scholion 2.** Prior sententia, qua materiae vis attractrix, tanquam proprietas essen[illegible] tribuitur, si esset vera, hoc commodi afferret, ut ulteriori investigatione causae liberaremur[illegible]

naturae scrutatoribus hoc onus gravissimum esset impositura, cui expediendo vires nunquam sufficerent. Ex quo ad nostrum commodum utique esset optandum, ut veritati foret consentanea. Multis autem premitur difficultatibus, quas cum primis nostrae principiis minime conciliare licet: si enim corpus ab alio attractum moveri incipit, non in illo sed in hoc altero est ponenda, quod cum ab illo sit remotum, concedendum corpus quaqua versus a nullis aliis corporibus cinctum, sed quasi in vacuo posi- moveri incipiat, etiamsi nusquam tangatur; neque tamen motus causam in ipso, sed in corpore longissime ab eo remoto esse quaerendam, hocque perinde evenire sive spatium sit vacuum, sive plenum. Corpus ergo, quatenus vi attrahendi esset praeditum, ad corpora utrumque remota quasi vires emitteret, quibus alia quasi comprehenderet et ad Quam actionem in distans qui concoquere non possunt, cum nullo modo concipi queat, quandam subtilem confugiunt, quae pressione in corpora agens ea phaenomena producat, attractioni, tanquam essentiali omnium corporum proprietati, adscribunt, etiamsi et illi hoc efficitur, explicare haud valeant.

Scholion 3. Hactenus saltem alium modum, quo duo corpora in se invicem agant, non nisi quando in se mutuo impetum faciunt et ad contactum perveniunt; cum enim tum in statu suo perseverare annitatur, hoc autem fieri nequeat, nisi se mutuo penetrent, eorum in causa est quominus hoc eveniat. Ex ea ergo nascantur, necesse est, vires utriusque immutantes, ut penetratio evitetur. Experientia etiam testatur, hoc casu majores vires non exeri, quam quae penetrationem avertere valeant. Istarum igitur virium, quarum ef- in conflictu corporum cernitur, origo in eorum impenetrabilitate manifesto est posita, unde percipere licet, quomodo duo corpora remota in se invicem agere valeant; ac si quis corpora ab aliis quantumvis remotis sine adminiculo medii inter ea existentis affici posse, nostrae principia funditus everti videntur, neque apparet quo jure tum siderum superstitionis commenta negari queant. Qui quidem a Leibnizii Harmonia praesta- abhorrent, concedere coguntur spirituum actioni corpora esse subjecta, quod etiam Leibniziani supremo negare non audent; spirituum autem actio in corpora ab omni contactu certe est statuenda, id quod attractioni favere videtur. Verum si corpora a spiritu concitantur, contactus, saltem praesentia quaedam concipi debet, ita ut etiam hinc actio corporum in firmamentum recipiat. Quare qui dicunt corporibus a Deo vim alia corpora quan- attrahendi tribui potuisse, nihil aliud dicere videntur, nisi Deum perpetuo immediate ad se invicem impellere. Verum omissa hac disputatione, ad Physicam potius quam huc certum sit singula materiae elementa perinde ad se mutuo impelli, ac si se attra- videamus, quales inde vires pro corporibus finitae magnitudinis nascantur.

Theorema. Corpus rigidum a viribus, quibus singula ejus elementa se mutuo attra- ad motum neutiquam sollicitatur.

Demonstratio. Veritas hujus theorematis isto nititur fundamento, quod vires, quibus duo mutuo attrahunt, utrinque sint aequales. Si enim duo concipiantur elementa, quorum massulae sint μ et ν, distantia vero $= z$, vis, qua μ ad ν attrahitur est $= \frac{C\mu\nu}{zz}$ (26), ac vi vis, qua ν ad μ attrahitur est $= \frac{C\mu\nu}{zz}$. Sunt igitur hae vires aequales, et quia alterum ad alt urgetur, earum directiones sunt contrariae. Jam in corpore elementum quodcunque ad reliqua attrahitur, et sumtis binis quibusque, vires, quibus alterum ab altero attrahitur, sunt aeq et contrariae, ideoque in corpore rigido se mutuo destruunt. Quod cum eveniat, quaecu bina elementa considerentur, necesse est omnes vires elementares, quibus cuncta elementa mutuo agunt, se mutuo destruere, propterea quod quaelibet vis habeat in corpore sibi aeq et contrariam.

33. **Coroll. 1.** Cum in hac mutua virium elementarium destructione ratio distantiae z n censum veniat, patet theorema fore verum, etiamsi attractio aliam quamcunque distantiae rati sequeretur, dummodo vires, quibus duo elementa se mutuo attrahunt, utrinque fuerint aequales

34. **Coroll. 2.** Sive ergo corpus rigidum quiescat, sive moveatur, a viribus, quibus elementa se mutuo appetunt, neque ejus status quietis, neque motus perturbabitur, sed a vi externis aeque afficietur, ac si elementa ejus vi attractrice carerent.

35. **Coroll. 3.** Quae ergo in Mechanica de motu corporum rigidorum traduntur, ea o veritati consentanea manent, etiamsi elementorum mutua attractio, a qua quidem ibi animum straxeramus, accedat, neque quidquam ibi propterea erit immutandum.

36. **Scholion 1.** Quod ad formulam $\frac{\mu\nu}{zz}$ attinet, cui vis, qua massula μ aliam massul in distantia $= z$ remotam attrahit, est proportionalis, quatenus ea constat ex reciproco qua distantiae et massa corpusculi attracti ν, ejus veritatem per phaenomena stabilivimus. Quatenus tem ea vis ipsi massae corpusculi attrahentis μ est proportionalis, id quidem sola ratione collegi Nunc igitur haec postrema ratio multo fortius corroboratur: Cum enim phaenomena etiam evi motum cujusque corporis coelestis non perturbari a viribus, quibus ejus partes se mutuo attra hinc vicissim intelligitur, vires, quibus bina quaeque elementa se mutuo attrahunt, aequales oportere, quoniam alioquin evenire posset, ut hae vires elementares se mutuo non destruerent. hoc clarius perspiciatur, sumamus vim attractricem non ipsi massae μ corpusculi attrahentis ejus quadrato μ^2 esse proportionalem, corpusque rigidum tantum ex duobus elementis μ et ν vallo z dissitis esse conflatum. Quo posito erit vis, qua ν ad μ attrahitur $= \frac{C\mu^2\nu}{zz}$, contra ver qua μ ad ν attrahitur $= \frac{C\mu\nu^2}{zz}$, quae ergo duae vires non essent aequales, nisi elementa aequalia, ideoque corpus totum excessu virium $\frac{C\mu\nu}{zz}(\mu - \nu)$ ad motum sollicitaretur, quod eo esset absurdum, cum idem corpus infinitis modis in duas partes dissectum concipi queat, et libet sectio peculiarem vim esset exhibitura, cujus etiam directio a sectionis ratione penden foret certa. Quod cum sit maxime absurdum, extra omne dubium est positum, vim attra cujusque elementi ipsius massae esse proportionalem, hancque rationis $\frac{\mu\nu}{zz}$ partem adeo multo esse evictam quam reliquas, cum hae tantum ex phaenomenis sint conclusae, illa autem adeo cipio contradictionis innitatur.

97. **Scholion 2.** Hoc theorema quidem tantum ad corpora rigida, quorum partes ita firmo nexu inter se sunt conjuncta, ut a nullis viribus de situ suo relativo dimoveri queant, accommodavimus, sed etiam quodammodo ad corpora flexibilia atque etiam fluida extenditur, quatenus scilicet tantum ad motum progressivum centri inertiae spectamus. Quodsi enim elementa corporis fuerint a se invicem dissoluta, quoniam vires, quibus bina quaeque se mutuo attrahunt, sunt aequales et contrariae, hinc nulla vis in centrum inertiae resultat. Ab his scilicet viribus fieri potest, ut partes quidem corporis quomodocunque inter se commoveantur, commune autem centrum inertiae jugiter quiescat, vel si semel moveri coeperit, perpetuo uniformiter in linea recta progrediatur. Ex quo efficitur, ut quotcunque fuerint corpora se mutuo attrahentia, eorum commune centrum inertiae vel quiescat, vel uniformiter in directum promoveatur. Hinc ergo certum est omnium corporum mundanorum commune centrum inertiae vel in perpetua quiete versari, vel uniformiter in directum [illegible] Atque hoc jam statim in ipso limine certissime affirmare licet, etiamsi adhuc minime pateat, quo motu singula corpora concitentur, in quo sine dubio veritas maximi momenti continetur.

98. **Problema 1.** Si corpus finitum, data figura praeditum, attrahat corpusculum ad datam et insignem ab eo distantiam remotum, definire tam quantitatem quam directionem ejus vis, qua corpusculum sollicitatur.

Solutio. (Fig. 172.) Cum corpusculum a singulis elementis corporis finiti attrahatur, definiri oportet vim ex omnibus istis viribus elementaribus resultantem. Quoniam igitur corporis finiti figura est data, tam ejus centrum inertiae quam ternos axes principales, eorumque respectu momenta inertiae pro cognitis assumi licet. Sit igitur J centrum inertiae corporis attrahentis, et JA, JB, JC ejus tres axes principales, eorumque respectu momenta inertiae Maa, Mbb, Mcc, denotante M corporis massam. Corpusculum autem attractum, cujus massa $= m$, sit in H in distantia ab illius centro inertiae $JH = h$, quae recta cum axibus principalibus faciat angulos $HJA = \alpha$, $HJB = \beta$ et $HJC = \gamma$, ut sit $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Hinc demisso ab H ad planum AJB perpendiculo HG, et ex G ad JA productam normali GF, erit $JF = h\cos\alpha$, $FG = h\cos\beta$ et $GH = h\cos\gamma$. Concipiatur jam corporis elementum quodcunque in Z, pro quo coordinatae axibus principalibus parallelae sint $JX = x$, $XY = y$, $YZ = z$; massa autem istius elementi sit $= dM$, eritque ex indole axium principalium et centri inertiae $\int x dM = 0$, $\int y dM = 0$, $\int z dM = 0$; porro $\int xy dM = 0$, $\int xz dM = 0$, $\int yz dM = 0$, atque $\int xx dM = \frac{1}{2}M(bb + cc - aa)$, $\int yy dM = \frac{1}{2}M(aa + cc - bb)$, $\int zz dM = \frac{1}{2}M(aa + bb - cc)$. Ducatur recta HZ, qua posita $= v$ erit

$$v = \sqrt{((h\cos\alpha - x)^2 + (h\cos\beta - y)^2 + (h\cos\gamma - z)^2)}, \quad \text{seu}$$

$$v = \sqrt{(hh - 2h(x\cos\alpha + y\cos\beta + z\cos\gamma) + x^2 + y^2 + z^2)}.$$

His positis, corpusculum m in H ad elementum dM in Z situm attrahitur vi $= \frac{CmdM}{vv}$, ubi quidem coefficientem constantem C negligere licet, deinceps, ubi opus fuerit, facile introducendum; ita ut vis haec secundum HZ sit $= \frac{mdM}{vv}$, quae resoluta secundum directiones $H\alpha$, $H\beta$, $H\gamma$ axibus principalibus parallelas, praebebit

$$\text{vim secundum } H\alpha = \frac{m(h\cos\alpha - x)\,dM}{v^3},$$

$$\text{vim secundum } H\beta = \frac{m(h\cos\beta - y)\,dM}{v^3},$$

$$\text{vim secundum } H\gamma = \frac{m(h\cos\gamma - z)\,dM}{v^3}.$$

Hinc ergo integrando vis tota, qua corpusculum m in H sollicitatur, componitur ex his tribus

$$\text{vi secundum } H\alpha = m\int\frac{(h\cos\alpha - x)\,dM}{v^3},$$

$$\text{vi secundum } H\beta = m\int\frac{(h\cos\beta - y)\,dM}{v^3},$$

$$\text{vi secundum } H\gamma = m\int\frac{(h\cos\gamma - z)\,dM}{v^3},$$

quae formulae ita generaliter ulterius tractari nequeunt. Sed quia distantia corpusculi $JH = h$ prae magnitudine corporis praegrandis supponitur, ita ut maximi valores, quos coordinatae x, y, z recipere possunt, prae h sint satis exigui, ejusmodi approximatione uti, qua valor ipsius v in seriem resolvatur, in qua coordinatarum x, y, z potestates, secunda altior, rejiciantur; hinc ergo fiet

$$\frac{1}{v^3} = \frac{1}{h^3} + \frac{3(x\cos\alpha + y\cos\beta + z\cos\gamma)}{h^4} - \frac{3(xx+yy+zz)}{2h^5} + \frac{15(x\cos\alpha + y\cos\beta + z\cos\gamma)^2}{2h^5},$$

unde colligitur pariter non ultra secundam potestatem ascendendo

$$\frac{h\cos\alpha - x}{v^3} = \frac{\cos\alpha}{hh} - \frac{x}{h^3}(1 - 3\cos^2\alpha) + \frac{3y\cos\alpha\cos\beta}{h^3} + \frac{3z\cos\alpha\cos\gamma}{h^3}$$

$$- \frac{3xx\cos\alpha}{2h^4}(3 - 5\cos^2\alpha) - \frac{3yy\cos\alpha}{2h^4}(1 - 5\cos^2\beta) - \frac{3zz\cos\alpha}{2h^4}(1 - 5\cos^2\gamma)$$

$$- \frac{3xy\cos\beta}{h^4}(1 - 5\cos^2\alpha) - \frac{3xz\cos\gamma}{h^4}(1 - 5\cos^2\alpha) + \frac{15yz\cos\alpha\cos\beta\cos\gamma}{h^4}.$$

Multiplicetur haec formula per DM, captisque singulorum membrorum integralibus secundum praecepta exposita, habebimus

$$\int\frac{(h\cos\alpha - x)}{v^3}dM = \frac{M\cos\alpha}{hh} - \frac{3M(bb+cc-aa)}{4h^4}\cos\alpha\,(3 - 5\cos^2\alpha)$$

$$- \frac{3M(aa+cc-bb)}{4h^4}\cos\alpha\,(1 - 5\cos^2\beta) - \frac{3M(aa+bb-cc)}{4h^4}\cos\alpha\,(1 - 5\cos^2\gamma)$$

quae forma ob $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ ad sequentem revocatur

$$\int\frac{(h\cos\alpha - x)\,dM}{v^3} = \frac{M\cos\alpha}{hh} + \frac{3Maa\cos\alpha}{2h^4}(3 - 5\cos^2\alpha)$$

$$+ \frac{3Mbb\cos\alpha}{2h^4}(1 - 5\cos^2\beta) + \frac{3Mcc\cos\alpha}{2h^4}(1 - 5\cos^2\gamma).$$

Si simili modo reliqua duo integralia colligantur, erunt ternae vires, quibus corpusculum m a corpore finito sollicitatur, sequentes:

$$\text{secundum}\quad H\alpha = \frac{Mm\cos\alpha}{hh}\Big(1 + \frac{3aa}{2hh}(3 - 5\cos^2\alpha) + \frac{3bb}{2hh}(1 - 5\cos^2\beta) + \frac{3cc}{2hh}(1 - 5\cos^2\gamma)\Big),$$

$$H\beta = \frac{Mm\cos\beta}{hh}\Big(1 + \frac{3bb}{2hh}(3 - 5\cos^2\beta) + \frac{3cc}{2hh}(1 - 5\cos^2\gamma) + \frac{3aa}{2hh}(1 - 5\cos^2\alpha)\Big),$$

$$H\gamma = \frac{Mm\cos\gamma}{hh}\Big(1 + \frac{3cc}{2hh}(3 - 5\cos^2\gamma) + \frac{3aa}{2hh}(1 - 5\cos^2\alpha) + \frac{3bb}{2hh}(1 - 5\cos^2\beta)\Big),$$

quae vires jam pro lubitu ad alias directiones reduci possunt.

69. **Coroll. 1.** Si harum trium virium quadrata colligantur, et ex summa radix quadrata extrahatur, prodit vis illis aequivalens, quae ergo simili approximatione adhibita reperitur:

$$\text{Vis aequivalens} = \frac{Mm}{hh}\Big(1 + \frac{3aa}{2hh}(1 - 3\cos^2\alpha) + \frac{3bb}{2hh}(1 - 3\cos^2\beta) + \frac{3cc}{2hh}(1 - 3\cos^2\gamma)\Big),$$

cujus directio deflectet a directione HJ angulo exiguo, qui est

$$= \frac{3}{hh}\sqrt{\Big((aa - bb)^2\cos^2\alpha\cos^2\beta + (aa - cc)^2\cos^2\alpha\cos^2\gamma + (bb - cc)^2\cos^2\beta\cos^2\gamma\Big)}.$$

70. **Coroll. 2.** Quodsi ergo corporis attrahentis terna momenta principalia fuerint inter se aequalia, $aa = bb = cc$, ob $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, erit vis, qua corpusculum H sollicitatur, $= \frac{Mm}{hh}$, ejusque directio in ipsam lineam HJ cadit, ita ut hoc casu corpusculum m in H perinde attrahatur, ac si tota corporis attrahentis massa M in ipso centro inertiae J esset collecta.

71. **Coroll. 3.** Si corpusculum attractum in axe principali corporis attrahentis JA fuerit situm, ut sit $\alpha = 0$ et $\beta = \gamma = 90^\circ$, deflexio vis attrahentis a directione HJ evanescit, ipsa vero vis attrahens erit $= \frac{Mm}{hh}\Big(1 + \frac{3\,(bb + cc - 2aa)}{2hh}\Big)$. Nisi ergo sit $bb + cc = 2aa$, attractio erit vel major vel minor, quam si tota corporis attrahentis massa in suo centro inertiae esset collecta.

72. **Coroll. 4.** Si corpus attractum ita fuerit situm, ut recta HJ ad singulos axes principales aeque inclinetur, angulo scilicet $54^0\,45'$, tum vis attrahens erit $= \frac{Mm}{hh}$, perinde ac si massa attrahens M in centro inertiae J esset collecta, at ejus directio declinabit a directione HJ angulo, qui est

$$= \frac{1}{hh}\sqrt{2\,(a^4 + b^4 + c^4 - aabb - aacc - bbcc)}.$$

73. **Scholion 1.** Vis, qua corpusculum m a corpore M attrahitur, infinitis aliis modis repraesentari potest. Primo scilicet haec vis aequivalet vi secundum $HJ = \frac{Mm}{hh}$, et insuper his tribus viribus:

$$\text{secundum}\quad H\alpha = \frac{3Mm\cos\alpha}{2h^4}\Big(aa\,(3 - 5\cos^2\alpha) + bb\,(1 - 5\cos^2\beta) + cc\,(1 - 5\cos^2\gamma)\Big),$$

$$H\beta = \frac{3Mm\cos\beta}{2h^4}\Big(bb\,(3 - 5\cos^2\beta) + cc\,(1 - 5\cos^2\gamma) + aa\,(1 - 5\cos^2\alpha)\Big),$$

$$H\gamma = \frac{3Mm\cos\gamma}{2h^4}\Big(cc\,(3 - 5\cos^2\gamma) + aa\,(1 - 5\cos^2\alpha) + bb\,(1 - 5\cos^2\beta)\Big),$$

... prae prima sunt vehementer parvae.

Deinde ista vis etiam hoc modo referri potest, ut aequivaleat

$$\text{vi secund. } HJ = \frac{Mm}{hh}\left(1 + \frac{5aa}{2hh}(1 - 3\cos^2\alpha) + \frac{5bb}{2hh}(1 - 3\cos^2\beta) + \frac{5cc}{2hh}(1 - 3\cos^2\gamma)\right),$$

insuperque his tribus valde parvis

$$\text{vi sec. } H\alpha = \frac{Mm(2aa - bb - cc)\cos\alpha}{h^4},$$

$$\text{vi sec. } H\beta = \frac{Mm(2bb - aa - cc)\cos\beta}{h^4},$$

$$\text{vi sec. } H\gamma = \frac{Mm(2cc - aa - bb)\cos\gamma}{h^4}.$$

Tum vero etiam hoc modo, ut aequivaleat

$$\text{vi sec. } HJ = \frac{Mm}{hh}\left(1 + \frac{3aa}{2hh}(1 - 3\cos^2\alpha) + \frac{3bb}{2hh}(1 - 3\cos^2\beta) + \frac{3cc}{2hh}(1 - 3\cos^2\gamma)\right),$$

et insuper his exiguis tribus

$$\text{vi sec. } H\alpha = \frac{3Mm\cos\alpha}{h^4}\left((aa - bb)\cos^2\beta + (aa - cc)\cos^2\gamma\right),$$

$$\text{vi sec. } H\beta = \frac{3Mm\cos\beta}{h^4}\left((bb - aa)\cos^2\alpha + (bb - cc)\cos^2\gamma\right),$$

$$\text{vi sec. } H\gamma = \frac{3Mm\cos\gamma}{h^4}\left((cc - aa)\cos^2\alpha + (cc - bb)\cos^2\beta\right).$$

Hinc si corpusculum m in plano axium JA et JB reperiatur, ut sit $\gamma = 90^0$ et $\alpha + \beta = 90^0$, vis id sollicitans constabit primo

$$\text{vi sec. } HJ = \frac{Mm}{hh}\left(1 + \frac{3aa}{2hh}(1 - 3\cos^2\alpha) + \frac{3bb}{2hh}(1 - 3\sin^2\alpha) + \frac{3cc}{2hh}\right),$$

tum vero his duabus viribus

$$\text{vi sec. } H\alpha = \frac{3Mm(aa - bb)\cos\alpha\cos^2\beta}{h^4},$$

$$\text{vi sec. } H\beta = \frac{3Mm(bb - aa)\cos\beta\cos^2\alpha}{h^4}.$$

Unde si momenta principalia respectu axium JA et JB fuerint aequalia, hae duae postremae vires evanescunt, remanetque sola vis prior

$$\text{sec. } HJ = \frac{Mm}{hh}\left(1 + \frac{3(cc - aa)}{2hh}\right).$$

44. **Scholion 2.** Quemadmodum corpora coelestia tantopere a se invicem sunt remota, ut nostra approximatio solutionem perfectam praebere sit censenda, ita etiam commode usu venit, ut eorum momenta inertiae sint fere inter se aequalia, unde proxime perinde ad se attrahuntur, ac si universa eorum massa in ipsorum centro inertiae esset collecta, quo casu vis attrahens perpetuo ad centrum inertiae foret directa, et quadrato distantiae reciproce proportionalis, prorsus uti Newtonus

Sin autem notabilis inaequalitas inter momenta inertiae intercederet, vis attrahens ab hac simplici aberraret, ut motus corpusculi attracti nonnisi difficillime inde definiri possit, praecipue si distantia non fuerit adeo magna. Cum enim corpusculum attractum continuo situm suum respectu axium principalium mutet, in motu ejus non solum distantia h, sed etiam anguli α, β, γ erunt quantitates variabiles, viresque assignatae maximam in calculo parient difficultatem. Unico casu difficultas minueretur, quando scilicet corporis attrahentis duo momenta principalia inter se fuerint aequalia, corpusque attractum in horum binorum axium plano moveatur: tum enim vis attrahens perpetuo ad centrum inertiae corporis attrahentis dirigetur, atque per solam distantiam determinabitur; verumtamen duabus constabit partibus, quarum altera quadrato, altera vero biquadrato distantiae erit reciproce proportionalis, quippe quae ut vidimus est $= \frac{Mm}{hh} + \frac{3Mm(cc-aa)}{2h^4}$, ubi quidem ob duplicem rationem pars posterior prae priori est vehementer parva, primo scilicet, quod quantitas hh plurimum superet quantitates cc et aa, tum vero quod ista quadrata aa et cc sint proxime inter se aequalia. Quod quo clarius perspiciatur, sit corpus attrahens sphaeroides ellipticum homogeneum, genitum ex conversione ellipsis, cujus semiaxes sint A et C, circa axem $2C$, ita ut bini semiaxes principales JA et JB futuri sint $= A$, ac tertius $JC = C$. Cum igitur massa istius corporis sit $M = \frac{4}{3}\pi ACC$, erit momentum inertiae respectu axium JA et $JB = \frac{1}{5}M(AA+CC)$, et respectu axis $JC = \frac{2}{5}MAA$, unde pro nostra formula fiet $aa = bb = \frac{AA+CC}{5}$ et $cc = \frac{2AA}{5}$, hincque $cc - aa = \frac{1}{5}(AA - CC)$. Quare si corpus fuerit sphaeroides compressum, uti terra, erit $AA > CC$, aliudque corpusculum circa id in plano aequatoris moveatur, vis attrahens major erit quam $\frac{Mm}{hh}$, et excessus erit biquadrato distantiae reciproce proportionalis, tota vi existente $= \frac{Mm}{hh} + \frac{3Mm(AA-CC)}{10h^4}$. Unde si semiaxes A et C proxime fuerint aequales, pars posterior prae priori fere pro evanescente haberi potest.

45. **Scholion 3.** Cum corpus solis sit fere perfecte sphaericum, in viribus, quibus planetae ad solem urgentur, haec inaequalitas tuto negligi potest, id quod etiam de viribus, quibus planetae primarii se mutuo attrahunt, multo magis est tenendum, cum hae vires ipsae prae vi solis sint vehementer exiguae. In motu quidem lunae aberratio figurae terrae a sphaerica alicujus momenti esse posse videtur, cum ob lunae vicinitatem, tum vero quod terrae figura magis a sphaerica recedat quam solis. Imprimis autem quando motus satellitum Jovis ac Saturni scrutari lubuerit, hujus aberrationis rationem haberi conveniet, propterea quod figura Jovis non parum a sphaerica recedit, dum ratio semiaxium $A:C$ fere ut $11:10$ reputatur, ac vicinitas satellitum hanc aberrationem eo magis adauget. In Saturno autem praeter eandem rationem annulus in vi attrahente notabilem perturbationem generare debet. Si enim annulus tanquam pars corporis Saturni spectetur, hoc figuram sphaeroidis admodum compressi induere est censendum. His autem casibus commode evenit, ut satellites Jovis fere in plano aequatoris hujus planetae, satellites Saturni autem fere in plano annuli revolvantur, quod si secus eveniret, investigationem motus satellitum ne suscipere quidem liceret. Quemadmodum autem figura corporis attrahentis hujusmodi anomaliam in vi attrahente gignere valet, ita similis anomalia quoque ex figura corporis attracti resultat, id quod in sequente problemate plenius ostendemus.

46. **Problema 2.** (Fig. 173.) Si corpus finitum attrahatur ad punctum N valde remot ad corpus, cujus totam massam in eo puncto collectam concipere licet, invenir attractricem, qua illud corpus sollicitatur.

Solutio. Sit J centrum inertiae corporis attracti, et JA, JB, JC ejus axes principales rum respectu ejus momenta inertiae sint Maa, Mbb, Mcc, denotante littera M ejus massam. autem attrahentis centrum inertiae sit in N, cujus effectus ut perinde se habeat, ac si massa, quae sit $= N$, in puncto N esset collecta, hujus corporis momenta inertiae omnia aequalia sunt concipienda, quemadmodum ex § 40 intelligitur. Hoc posito, singula corporis elementa ad ipsum punctum N attrahantur viribus distantiarum quadratis reciproce proportio et quoniam hae vires aequales sunt et contrariae viribus, quibus corpusculum in N collocatum quidem massa esset $= N$, ad singula corporis M elementa attrahitur, vis etiam tota, qua a corpore N sollicitatur, aequalis et contraria erit vi, qua corpus N, ut punctum consid corpore M attrahitur, et quam in problemate praecedente determinavimus. Ponatur ergo inertiae distantia $JN = h$, et anguli $NJA = \alpha$, $NJB = \beta$, $NJC = \gamma$, ductisque ex N rectis $N\gamma$ parallelis ternis axibus principalibus JA, JB, JC corporis attracti, vires, quibus corpus sollicitatur, ita ad directiones $N\alpha$, $N\beta$, $N\gamma$ reducentur, ut sit vis

$$\text{sec. } N\alpha = \frac{MN\cos\alpha}{hh}\left(1 + \frac{3aa}{2hh}(3 - 5\cos^2\alpha) + \frac{3bb}{2hh}(1 - 5\cos^2\beta) + \frac{3cc}{2hh}(1 - 5\cos^2\gamma)\right)$$

$$\text{sec. } N\beta = \frac{MN\cos\beta}{hh}\left(1 + \frac{3bb}{2hh}(3 - 5\cos^2\beta) + \frac{3cc}{2hh}(1 - 5\cos^2\gamma) + \frac{3aa}{2hh}(1 - 5\cos^2\alpha)\right)$$

$$\text{sec. } N\gamma = \frac{MN\cos\gamma}{hh}\left(1 + \frac{3cc}{2hh}(3 - 5\cos^2\gamma) + \frac{3aa}{2hh}(1 - 5\cos^2\alpha) + \frac{3bb}{2hh}(1 - 5\cos^2\beta)\right)$$

quippe quae sunt aequales et contrariae iis, quas in problemate praecedente invenimus, nisi hic corporis attrahentis massa sit N, cum ibi ejus loco habuissemus litteram m. Etsi autem vires hic ad punctum N sint relatae, tamen corpus ABC sollicitare sunt censendae; scilicet oportet vim illis ternis aequivalentem, cujus directio ad corpus ABC usque producta vim hoc pus sollicitantem manifestabit. Potest autem ex his viribus una vis elici secundum direction sollicitans, quae quasi vis primaria spectari potest, prae qua reliquae sint valde parvae. Scili in § 43 vires corpus ABC sollicitantes ita repraesentari possunt, ut primo adsit

$$\text{vis sec. } JN = \frac{MN}{hh}\left(1 + \frac{5aa}{2hh}(1 - 3\cos^2\alpha) + \frac{5bb}{2hh}(1 - 3\cos^2\beta) + \frac{5cc}{2hh}(1 - 3\cos^2\gamma)\right);$$

tum vero hae tres vires valde exiguae

$$\text{sec. } N\alpha = \frac{MN(2aa - bb - cc)\cos\alpha}{h^4},$$

$$\text{sec. } N\beta = \frac{MN(2bb - aa - cc)\cos\beta}{h^4},$$

$$\text{sec. } N\gamma = \frac{MN(2cc - aa - bb)\cos\gamma}{h^4}.$$

... modo ut primo adsit vis principalis

$$\text{sec. } JN = \frac{MN}{hh}\left(1 + \frac{3aa}{2hh}(1 - 3\cos^2\alpha) + \frac{3bb}{2hh}(1 - 3\cos^2\beta) + \frac{3cc}{2hh}(1 - 3\cos^2\gamma)\right)$$

ac praeterea hae tres vires minimae

$$\text{sec. } N\alpha = \frac{3MN\cos\alpha}{h^4}\left((aa - bb)\cos^2\beta + (aa - cc)\cos^2\gamma\right),$$

$$\text{sec. } N\beta = \frac{3MN\cos\beta}{h^4}\left((bb - aa)\cos^2\alpha + (bb - cc)\cos^2\gamma\right),$$

$$\text{sec. } N\gamma = \frac{3MN\cos\gamma}{h^4}\left((cc - aa)\cos^2\alpha + (cc - bb)\cos^2\beta\right).$$

77. **Coroll. 1.** Si igitur et corporis ABC terna momenta principalia fuerint inter se aequalia, sola vis secundum $JN = \frac{MN}{hh}$ relinquitur, et ambo corpora utut finita perinde se attrahant, ac si utriusque massa in suo centro inertiae esset collecta, ac tum directio vis attractricis per utriusque centrum inertiae transit.

78. **Coroll. 2.** Si corpus attrahens N in plano AJB binis axibus principalibus JA et JB corporis attracti determinato versetur, erit $\gamma = 90^0$ et $\alpha + \beta = 90^0$; unde hoc casu habebitur

$$\text{vis sec. } JN = \frac{MN}{hh}\left(1 + \frac{3aa}{2hh}(1 - 3\cos^2\alpha) + \frac{3bb}{2hh}(1 - 3\sin^2\alpha) + \frac{3cc}{2hh}\right),$$

ac praeterea hae duae tantum vires valde parvae

$$\text{vis sec. } N\alpha = \frac{3MN(aa - bb)\cos\alpha\cos^2\beta}{h^4},$$

$$\text{vis sec. } N\beta = \frac{3MN(bb - aa)\cos\beta\cos^2\alpha}{h^4}.$$

79. **Coroll. 3.** Quare si hoc casu corporis attracti momenta inertiae respectu axium JA et JB fuerint inter se aequalia, binae vires exiguae secundum $N\alpha$ et $N\beta$ evanescunt, relinquiturque vis attrahens sola secundum $JN = \frac{MN}{hh}\left(1 + \frac{3(cc - aa)}{2hh}\right)$, quae ergo partim quadrato partim biquadrato distantiae est proportionalis.

80. **Coroll. 4.** In genere autem si fuerit $bb = aa$, seu corporis ABC omnia momenta inertiae respectu axium in plano AJB sumtorum sint aequalia, vis sollicitans constabit

$$\text{vi sec. } JN = \frac{MN}{hh}\left(1 + \frac{3(cc - aa)(1 - 3\cos^2\gamma)}{2hh}\right)$$

et insuper his viribus exiguis

$$\text{vi sec. } N\alpha = \frac{3MN(aa - cc)\cos\alpha\cos^2\gamma}{h^4},$$

$$\text{vi sec. } N\beta = \frac{3MN(aa - cc)\cos\beta\cos^2\gamma}{h^4},$$

$$\text{vi sec. } N\gamma = \frac{3MN(cc - aa)\cos\gamma\sin^2\gamma}{h^4}.$$

51. **Scholion 1.** Effectus ab his viribus oriundus duplici modo se habet, prout vel motum progressivum corporis ABC afficit, vel ejus motum gyratorium, quos binos effectus seorsim definire licet. Quodsi ergo ad solum motum progressivum corporis ABC respiciamus, singulas sollicitantes, tanquam ipsi ejus centro inertiae J in suis directionibus applicatas consideramus, propterea corpus tum ab his viribus sollicitatum est censendum, primo a

$$\text{vi sec. } JN = \frac{MN}{hh}\left(1 + \frac{3aa}{2hh}(1 - 3\cos^2\alpha) + \frac{3bb}{2hh}(1 - 3\cos^2\beta) + \frac{3cc}{2hh}(1 - 3\cos^2\gamma)\right)$$

ac praeterea ab his tribus exiguis viribus

$$\text{vi sec. } JA = \frac{3MN}{h^4}\cos\alpha\left((aa - bb)\cos^2\beta + (aa - cc)\cos^2\gamma\right),$$

$$\text{vi sec. } JB = \frac{3MN}{h^4}\cos\beta\left((bb - cc)\cos^2\gamma + (bb - aa)\cos^2\alpha\right),$$

$$\text{vi sec. } JC = \frac{3MN}{h^4}\cos\gamma\left((cc - aa)\cos^2\alpha + (cc - bb)\cos^2\beta\right).$$

Sin autem perturbationem motus vertiginis, quo corpus ABC circumagitur, definire velimus, tam ad ipsas vires, quam earum momenta respectu axium principalium respicere debemus. Ex viribus autem secundum directiones $N\alpha$, $N\beta$, $N\gamma$ agentibus colligimus primo momentum respectu JA in sensum BC tendens $=$ vi $N\gamma . FG$ — vi $N\beta . GN =$ vi $N\gamma . h\cos\beta$ — vi $N\beta . h\cos\gamma$. Cum ergo primaria secundum JN nullum praebeat momentum, ex viribus exiguis sequentia colligimus momenta

I. Momentum respectu axis JA in sensum $BC = \dfrac{3MN(cc - bb)\cos\beta\cos\gamma}{h^3}$,

II. Momentum respectu axis JB in sensum $CA = \dfrac{3MN(aa - cc)\cos\alpha\cos\gamma}{h^3}$,

III. Momentum respectu axis JC in sensum $AB = \dfrac{3MN(bb - aa)\cos\alpha\cos\beta}{h^3}$,

unde quemadmodum perturbatio motus gyratorii definiri debeat, in libro superiori abunde est ostensum.

52. **Scholion 2.** Si corpora coelestia essent perfecte sphaerica et ex materia homogenea conflata, seu saltem eorum momenta inertiae inter se aequalia, haud aliter se mutuo attraherent, ac si singulorum massae in suo quaeque centro inertiae essent collectae, quam hypothesin etiam Newtonus in investigatione motuum coelestium assumsit. Verum si corpora coelestia ab hac figura recedunt, iisdem principiis inducti videmus vires attractrices neque ad centrum inertiae cujusque corporis tendere, neque exacte quadratis distantiarum reciproce esse proportionales, ac nunc quidem intelligimus aberrationem ab hac lege a duplici causa proficisci, a figura scilicet non sphaerica tam corporis attrahentis quam corporis attracti. In primo nempe problemate in corpore attracto momenta inertiae aequalia, et in attrahente inaequalia; in secundo autem problemate in corpore attracto momenta inertiae inaequalia et in attrahente aequalia sumsimus; unde dum alterum corpus habeat momenta inertiae aequalia, inde vis attractrix determinari potest. Superesset igitur, ut investigaremus quoque casum, quo ambo corpora habeant sua momenta inertiae inaequalia; verum etiamsi hanc investigationem feliciter expediremus, vix quicquam lucri inde consequeremur, cum calculus pro motu

minando instituendus nimis difficilis redderetur. Quo etiam facile erit vim attrahentem his casibus proxime assignare, modo unum modo alterum corpus tanquam sphaericum spectando et aberrationes a lege vulgari colligendo. Sed plerumque hae aberrationes tam sunt exiguae pro motu progressivo, ut satis tuto negligi queant; pro motu autem vertiginis sufficit figuram corporis attracti considerasse, quoniam vires sollicitantes per praelongos vectes agunt, unde tanta earum momenta nascuntur, ut prae iis momenta virium, quae ex figura corporis attracti resultarent, pro nihilo essent habenda, quandoquidem hae in ipso corpore attracto applicatae sunt censendae, ideoque nonnisi exigua momenta producere valent.

53. **Scholion 3.** Hae ergo sunt vires, quibus corpora coelestia sollicitantur, et ex quarum actione eorum motus ex principiis mechanicis investigari oportet, qui, quin deinceps vicissim cum observationibus exactissime conveniant, eo minus est dubitandum, cum existentia harum virium sit per ipsas observationes confirmata. Cum igitur hae vires neque sint quadratis distantiarum reciproce proportionales, neque ad ipsa centra inertiae cujusque corporis tendant, nisi corpora sint sphaerica, seu omnia momenta inertiae habeant aequalia, in multo difficiliores investigationes delabimur, quam quidem Newtonus susceperat. Praeterea vero insigne dubium hic oritur, an hae vires solae in mundo existant, quibus corpora coelestia sollicitentur? Etsi enim phaenomena alias nobis non patefaciant, tamen cum universum spatium aethere, utpote luminis vehiculo sit repletum, fieri omnino nequit, quin inde resistentia quaedam nascatur, quam corpora coelestia in motu suo patiantur. Nam quantumvis etiam subtilem aetheris materiam fingamus, tamen neque liberrime corpora crassiora penetrare statui potest, neque ea usque adeo poris plena concipere licet, ut in motu suo nullos plane impetus ab aethere sustineant. Ob poros quidem aetheri pervios concedere debemus, ejus resistentiam prae ea, quam alia fluida objicerent, multo minorem esse quam pro ratione densitatis; scilicet si aether esset centies millies rarior aëre, resistentia multo magis quam centies millies minor esset statuenda prae ea, quam idem corpus pari celeritate motum in aëre sentiret. Si quis objicere vellet, aetherem cuique corpori coelesti vicinum pari celeritate proferri, ideoque nullam inde resistentiam oriri, quod quidem nulla ratione confirmari potest, is tamen concedere teneretur, cometas resistentiam pati debere. Quod autem effectum hujusmodi resistentiae non percipiamus, causa est, quod nonnisi post plura secula sensibilis evadat: ac tantum abest, ut observationes huic sententiae plane adversentur, ut potius in motu lunae talis effectus animadverti videatur.

54. **Theorema.** Quotcunque fuerint corpora, quae se mutuo attrahant, et quomodocunque moveantur, eorum commune centrum inertiae vel quiescit, vel uniformiter in directum promovetur.

Demonstratio. Veritas hujus theorematis isto nititur fundamento, quod vires, quibus duo quaeque corpora se invicem attrahunt, sint inter se aequales et in contrarium directae. Scilicet si fuerint (fig. 174) duo corpora quaecunque Ma et Nb, et vis, qua se mutuo attrahunt $= V$, cujus directio sit recta ab, ita ut corpus Ma a corpore Nb sollicitetur a vi $= V$ in directione ab, corpus vero Nb a corpore Ma vi $= V$ in directione ba. Considerentur utriusque corporis centra inertiae, quae sint in A et B, et quatenus tantum ad motum progressivum horum corporum respicimus, utrique corpori vis, qua sollicitatur, in ipso centro inertiae secundum eandem directionem appli... est concipienda. Hinc corpus A sollicitari censendum est vi $= V$ in directione $A\alpha$, et corpus ... vi $= V$ in directione $B\beta$, ita ut vires sint aequales et directiones parallelae oppositaeque. Quare ... motus ad ternas directiones fixas referatur, et vis corpus A sollicitans secundum has ternas di... rectiones resoluta praebeat vires P, Q, R, vis corpus B sollicitans eodem modo resoluta dabit ... $-P$, $-Q$, $-R$. His positis (fig. 175) sint corpora quotcunque se mutuo attrahentia, quorum ... massae, in cujusque centro inertiae A, B, C collectae, denotentur literis A, B, C, pro quibus ... coordinatae ternis directionibus fixis OE, OF, OG parallelae statuantur

$$O\alpha = x,\ \alpha a = y,\ aA = z;\quad O\beta = x',\ \beta b = y',\ bB = z';\quad O\gamma = x'',\ \gamma c = y'',\ cC = z''.$$

Sit porro vis, qua corpora A et B se mutuo attrahunt, $= V$, vis corporum A et $C = V'$ et ... corporum B et $C = V''$, quae secundum easdem directiones fixas resolutae dent vires P, Q, R ... P', Q', R', et P'', Q'', R'' atque

corpus	sollicitabitur secundum		
	direct. OE	direct. OF	direct. OG
A	$+P+P'$	$+Q+Q'$	$+R+R'$
B	$-P+P''$	$-Q+Q''$	$-R+R''$
C	$-P'-P''$	$-Q'-Q''$	$-R'-R''$

Jam ex principiis accelerationis, sumto elemento temporis dt constante, colligimus formulas sequentes

$$Addx = 2gdt^2(+P+P');\quad Addy = 2gdt^2(+Q+Q');\quad Addz = 2gdt^2(+R+R')$$
$$Bddx' = 2gdt^2(-P+P'');\quad Bddy' = 2gdt^2(-Q+Q'');\quad Bddz' = 2gdt^2(-R+R'')$$
$$Cddx'' = 2gdt^2(-P'-P'');\quad Cddy'' = 2gdt^2(-Q'-Q'');\quad Cddz'' = 2gdt^2(-R'-R'')$$

unde concludimus

$$Addx + Bddx' + Cddx'' = 0$$
$$Addy + Bddy' + Cddy'' = 0$$
$$Addz + Bddz' + Cddz'' = 0,$$

simulque patet si plura tribus fuerint corpora, hujusmodi ternas aequationes semper oriri debere. Hinc ergo integrando consequimur

$$Adx + Bdx' + Cdx'' = Edt;\quad Ax + Bx' + Cx'' = Et + \mathfrak{E}$$
$$Ady + Bdy' + Cdy'' = Fdt;\quad Ay + By' + Cy'' = Ft + \mathfrak{F}$$
$$Adz + Bdz' + Cdz'' = Gdt;\quad Az + Bz' + Cz'' = Gt + \mathfrak{G},$$

unde apparet, commune centrum inertiae corporum secundum singulas directiones OE, OF, OG uniformiter proferri, ideoque ejus motum verum fore uniformem in directum. Nisi igitur commune centrum inertiae quiescat, certe uniformiter in directum profertur.

55. **Coroll. 1.** Si igitur toti systemati motus aequalis et contrarius ei, quo commune centrum inertiae progreditur, imprimi concipiatur, corpora ita movebuntur, ut commune centrum inertiae in quiete persistat.

56. **Coroll. 2.** Hoc autem motu superaddito, motus corporum respectivus inter se non mutabitur, ex quo totius mundi commune centrum inertiae tanquam quiescens spectari potest, motus enim singulorum corporum ab iisdem viribus efficietur, sive istud centrum quiescat, sive uniformiter in directum promoveatur.

57. **Coroll. 3.** Quare si motum corporum respectivum, respectu communis centri inertiae definire velimus, non alias vires, praeter eas, quibus singula corpora revera sollicitantur, considerari opus est.

58. **Scholion 1.** In Astronomia autem neque motus corporum coelestium absolutos, neque ad eorum commune centrum inertiae relatos contemplari solemus, sed potius propositum est eorum motus respectu corporis cujuspiam mundani, quales spectatori, in ejus centro inertiae collocato, essent apparituri, assignare. Ita motus planetarum primariorum ut et cometarum ad centrum solis, secundariorum vero ad centrum primarii referri solent. Ut autem hi motus apparentes per calculum reperiantur, id corpus, ex cujus centro ii spectari concipiuntur, tanquam quiescens assumitur, reliquis vero, praeter vires, quibus revera sollicitantur, insuper vires applicari debent, quae sint similes et contrariae iis, quibus corpus quiescens urgetur; scilicet hae vires primo per massam corporis quiescentis dividi, tum vero iterum per massam cujusque corporis moti, cui sunt applicandae, multiplicari debent, ut in his corporibus aequalem et contrariam motus perturbationem producant ei, quam in corpore, quod quiescens assumitur, producturae fuissent. Haec regula rite observata perducet ad motus apparentes, ex quibus deinceps motus veri, si motus corporis, quod pro quiescente assumitur, fuerit cognitus, facile definiuntur, siquidem opus fuerit eos nosse: semper enim sufficit motus tantum respectivos inter se habere cognitos. Ac si hos motus respectu unius noverimus, facile respectu cujusque alius determinabimus, quemadmodum Astronomi ex locis planetarum heliocentricis eorum loca geometrica elicere solent: scilicet cognito motu planetarum ac proinde etiam terrae, qualis ex centro solis esset apparituros, per solam Geometriam inde colligitur motus eorum, quo spectatori in centro terrae posito progredi videntur, unde deinceps etiam motus apparens pro spectatoribus in superficie terrae constitutis facile concluditur.

59. **Scholion 2.** Cum numerus corporum in mundo existentium et se mutuo attrahentium sit quasi infinitus, eorum motus exactissime definire haud licet, nisi pro corporum se mutuo attrahentium numero quantumvis magno motus inde oriundos determinare voluerimus, quod problema tantis calculi difficultatibus involutum deprehenditur, ut sagacitas humana illi enodando minime sufficere videatur. Verum corpora mundana ita commode a sapientissimo Conditore disposita deprehenduntur, ut primo sol et stellae fixae tam vastis a se invicem intervallis sint remotae, ut vires, quibus in se mutuo agunt, non obstante horum corporum insigni mole, pro nihilo sint reputandae; unde fit, ut stellas fixas cum sole tanquam corpora quiescentia et nullis viribus se mutuo sollicitantia spectare liceat, quod commodum neutiquam locum esset habiturum, si minoribus intervallis a se invicem distarent. Deinde etiam planetae et cometae nunquam tantopere a sole recedere videntur, ut vires, quas a stellis fixis sustinent, prae vi solis quicquam momenti adipiscantur. Tum vero etiam planetae principales et cometae pro suis massis tantis distantiis a se invicem manent sejuncti, ut vires, qui-

bus se mutuo afficiunt, prae viribus, quibus ad solem tendunt, sint satis exiguae. Luna autem satellites Jovis et Saturni tam vicini sunt suis principalibus, ut vires, quibus ad eos urgentur, plurimum excedant ipsam vim solis. Quare pro motu omnium horum corporum proxime determinando sufficit unicam vim considerasse, dum reliquae prae ea sint valde parvae, quarum effectus tantum in exiguis perturbationibus producendis consumuntur, quas ope methodi approximandi definire licet. Si autem vel planetae primarii sibi essent multo propiores, vel satellites a suis principalibus magis distarent, nullo fere modo ad motus eorum cognitionem pertingere possemus.

60. **Scholion 3.** Primo ergo duo tantum corpora se mutuo attrahentia contemplari conveniet, ubi quidem eorum indoles, prout fuerint sphaerica vel non sphaerica, investigationem bipartitam reddet. Nomine autem corporum sphaericorum complector omnia ea, in quibus terna momenta principalia sunt aequalia, reliqua omnia non sphaerica appellans. Sphaeroidica autem corpora in genere mihi erunt ea, in quibus duo momentorum principalia sunt aequalia, quae ergo unico axe principali sunt praedita, dum bini reliqui fuerint indefiniti, atque ad hoc genus omnia corpora coelestia referenda videntur. Expedito autem binorum corporum motu, ad terna progrediamur, quousque scilicet licuerit. Si enim problema in genere resolvere nequeamus, contenti esse poterimus approximationibus inde petitis, quod prae una vi reliquae sint valde exiguae, qui casus in mundo ubique locum habere videtur. Denique quid aetheris resistentia valeat erit inquirendum, ac tandem perturbatio in motu vertiginis a momentis virium sollicitantium oriunda Astronomiae mechanicae finem imponet.

Caput II.

De motu duorum corporum sphaericorum se mutuo attrahentium.

61. **Problema.** Si duo corpora sphaerica se mutuo attrahant, definire motum alterius, qualis spectatori in alterius centro posito est appariturus, ad planum, in quo ipse motus absolvitur, relatum.

Solutio. (Fig. 176.) Sint A et B duo corpora, quae litterae simul eorum massas denotent, et observator constitutus sit in centro corporis A, quod propterea ut quiescens consideretur. Jam quia virium, quibus se mutuo attrahunt, directio per utriusque centrum transit, quomodocunque corpus B moveri coeperit, directio vis sollicitantis semper est in plano per motus directionem et centrum corporis A transeunte, ideoque corpus B in eodem plano progredi perget. Quare tabula repraesentet hoc planum, in quo centrum corporis B moveri videtur, et cum initio ex E fuerit egressum, elapso tempore t pervenerit in B, ita ut circa A confecerit angulum $EAB = \varphi$, sitque distantia $AB = v$, unde patet si ad quodvis tempus t tam angulum $EAB = \varphi$ quam distantiam $AB = v$ assignare potuerimus, motum corporis B perfecte fore cognitum. Cum igitur B trahatur ad A in directione BA vi $= \frac{A}{vv}$, parique vi corpus A ad B in directione AB sollicitetur, haec posterior vis in corpus B translata fiet $= \frac{B}{vv}$, idque in directione AB afficere censendum est, corpus B sollicitetur omnino vi $=\frac{B(A+B)}{vv}$ in directione BA, quoniam corpus A ut quiescentem consideramus. Ex B in directionem fixam AE demisso perpendiculo BX, ut sit $AX = v\cos\varphi$ et $XB = v\sin\varphi$, et secundum easdem directiones vis $BA = \frac{B(A+B)}{vv}$ resolvatur, erit vis secundum $XA = \frac{B(A+B)}{vv}\cos\varphi$, et vis secundum $BX = \frac{B(A+B)}{vv}\sin\varphi$. Hinc sumto elemento temporis dt constante, ex principiis Mechanicae elicimus has binas aequationes:

$$dd.v\cos\varphi = \frac{-2g(A+B)}{vv}dt^2\cos\varphi; \qquad dd.v\sin\varphi = \frac{-2g(A+B)}{vv}dt^2\sin\varphi,$$

ubi g est altitudo, per quam grave delabitur tempore unius minuti secundi, siquidem tempus t detur in minutis secundis; at hic formula virium per certam constantem multiplicari oporteret, cujus magnitudo ex dato casu esset definienda. Verum hanc ipsam constantem sine ulla confusione subintelligere licet. En ergo has duas aequationes solutionem problematis continentes:

$$\text{I.} \quad ddv\cos\varphi - 2dv\,d\varphi\sin\varphi - vd\varphi^2\cos\varphi - vdd\varphi\sin\varphi = \frac{-2g(A+B)}{vv}dt^2\cos\varphi,$$

$$\text{II.} \quad ddv\sin\varphi + 2dv\,d\varphi\cos\varphi - vd\varphi^2\sin\varphi + vdd\varphi\cos\varphi = \frac{-2g(A+B)}{vv}dt^2\sin\varphi,$$

unde haec combinatio $\text{II}.\cos\varphi - \text{I}.\sin\varphi$ praebet

$$2dv\,d\varphi + vdd\varphi = 0,$$

quae per v multiplicata dat hoc integrale

$$vvd\varphi = Cdt \quad \text{hincque} \quad d\varphi = \frac{Cdt}{vv}.$$

At prima aequatio per v multiplicata ita repraesentari potest:

$$vddv\cos\varphi - d.vvd\varphi\sin\varphi = \frac{-2g(A+B)}{v}dt^2\cos\varphi,$$

ubi, ob $vvd\varphi = Cdt$, est $d.vvd\varphi\sin\varphi = Cdtd\varphi\cos\varphi$, ideoque

$$vddv - Cdtd\varphi + \frac{2g(A+B)}{v}dt^2 = 0, \quad \text{seu}$$

$$vddv - \frac{CCdt^2}{vv} + \frac{2g(A+B)}{v}dt^2 = 0,$$

quae multiplicata per $\frac{2dv}{v}$ praebet

$$2dvddv - \frac{2CCdt^2dv}{v^3} + \frac{4g(A+B)dt^2}{vv} = 0,$$

cujus integrale est

$$dv^2 + \frac{CCdt^2}{vv} - \frac{4g(A+B)dt^2}{v} = Ddt^2,$$

unde elicitur

$$dt = \frac{vdv}{\sqrt{(Dvv + 4g(A+B)v - CC)}}$$

hincque

$$d\varphi = \frac{Cdv}{v\sqrt{(Dvv + 4g(A+B)v - CC)}}.$$

Cum igitur hinc per v definiantur tam t et φ, vicissim pro dato tempore t assignare licebit variabilium v et φ.

62. **Coroll. 1.** Prima aequatio integralis $vvd\varphi = Cdt$ continet elementum areae desc... BAb, quod est $= \frac{1}{2} vvd\varphi$, unde tota area $EAB = \frac{1}{2} \int vvd\varphi$ aequalis fit $\frac{1}{2}\, Ct$, ideoque tempori proportionalis.

63. **Coroll. 2.** Aequatio inter v et φ inventa

$$d\varphi = \frac{Cdv}{v\sqrt{(Dvv + 4g(A+B)v - CC)}}$$

exprimit naturam curvae EB, quam corpus B circa A describere videtur. Eam autem esse sectionem conicam mox ostendemus.

64. **Coroll. 3.** Cum $-CC$ necessario sit quantitas negativa, ex formula irrationali

$$\sqrt{(Dvv + 4g(A+B)v - CC)}$$

patet distantiam v evanescere nunquam posse, nisi sit $C = 0$, quo casu ob $d\varphi = 0$, corpus B in linea recta ad A esset accessurum.

65. **Coroll. 4.** At si non est $C = 0$, necesse est, ut distantia v semper limitem quendam superet, qui limes, si constans D sit positiva, est

$$= \frac{\sqrt{(4gg(A+B)^2 + CCD)} - 2g(A+B)}{D}.$$

Sin autem D sit quantitas negativa $= -E$, erit limes

$$= \frac{2g(A+B) - \sqrt{(4gg(A+B)^2 - CCE)}}{E};$$

at si $D = 0$, limes iste fit $= \frac{CC}{4g(A+B)}$.

Resolutio formularum.

66. Quoniam distantia v superare debet certum limitem, si hic ponatur $= h$, erit $v - h$ factor formulae post signum radicale $Dvv + 4g(A+B)v - CC$, et alter factor erit formae $k \pm v$, prout D fuerit vel positivum vel negativum. Commodius autem scopum attingemus ponendo $v = \frac{f}{u}$, unde ob $dv = \frac{-fdu}{uu}$, erit

$$dt = \frac{-ffdu}{uu\sqrt{(Dff + 4fg(A+B)u - CCuu)}}$$

$$\text{et} \quad d\varphi = \frac{-Cdu}{\sqrt{(Dff + 4fg(A+B)u - CCuu)}}.$$

Hic si ponatur $Cu = Cp + \frac{2fg(A+B)}{C}$, fit formula radicalis $= \sqrt{(Dff + \frac{4ffgg(A+B)^2}{CC} - CCpp)}$, et ob $-Cdu = -Cdp$, integrale posterioris aequationis erit

$$\alpha + \varphi = \text{Arc.}\cos.\frac{CCp}{\sqrt{(CCDff + 4ffgg(A+B)^2)}}.$$

Habebimus ergo

$$p = \frac{f\cos(\alpha+\varphi)}{CC}\sqrt{(CCD + 4gg(A+B)^2)}$$

$$\text{et} \quad u = \frac{2fg(A+B)}{CC} + \frac{f\cos(\alpha+\varphi)}{CC}\sqrt{(CCD + 4gg(A+B)^2)}.$$

Quo has formulas commodiores reddamus, constantes ita definiamus, ut fiat $u = 1 + n\cos s$, eritque

$$\alpha + \varphi = s, \quad \frac{2fg(A+B)}{CC} = 1 \quad \text{et} \quad \frac{f}{CC}\sqrt{(CCD + 4gg(A+B)^2)} = n.$$

Quare ob $CC = 2fg(A+B)$, sumtis quadratis habebitur

$$\frac{Df}{2g(A+B)} + 1 = nn \quad \text{et} \quad D = \frac{-2(1-nn)g(A+B)}{f}, \quad \text{seu} \quad D = \frac{-(1-nn)CC}{ff}.$$

Hinc pro signo radicali obtinebimus

$$\sqrt{(-CC + nnCC + 2CCu - CCuu)} = C\sqrt{(nn-(1-u)^2)},$$

quae, ob $u - 1 = n\cos s$ abit in $Cn\sin s$, ubi meminisse oportet esse $C = \sqrt{2fg(A+B)}$. Cum ergo sit

$$v = \frac{f}{1+n\cos s} \quad \text{et} \quad d\varphi = \frac{Cnds\sin s}{Cn\sin s} = ds, \text{ erit } \varphi = s + \text{Const.} \quad \text{et} \quad Cdt = \frac{ffds}{(1+n\cos s)^2},$$

unde elicimus

$$\frac{Ct}{ff} = \frac{1}{1-nn}\int\frac{ds}{1+n\cos s} - \frac{n\sin s}{(1-nn)(1+n\cos s)}.$$

Jam vero si $n < 1$ est

$$\int\frac{ds}{1+n\cos s} = \frac{1}{\sqrt{(1-nn)}}\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s};$$

sin autem $n > 1$ erit

$$\int\frac{ds}{1+n\cos s} = \frac{1}{\sqrt{(nn-1)}}\log\frac{n+\cos s+\sin s\sqrt{(nn-1)}}{1+n\cos s}.$$

Casu autem quo $n = 1$ reperitur

$$\frac{Ct}{ff} = \int\frac{ds}{(1+\cos s)^2} = \frac{(2+\cos s)\sin s}{3(1+\cos s)^2}.$$

In hac ergo resolutione loco binarum constantium C et D aliae duae f et n introducuntur, et omnia per novam variabilem, angulum scilicet s, ita definiuntur ut sit

$$\text{I. } \varphi = s + \text{Const.} \quad \text{II. } v = \frac{f}{1+n\cos s} \quad \text{et} \quad dt\sqrt{2fg(A+B)} = \frac{ffds}{(1+n\cos s)^2}, \quad \text{sive}$$

$$\text{III. } t = \frac{f\sqrt{f}}{\sqrt{2g(A+B)}}\int\frac{ds}{(1+n\cos s)^2},$$

cujus solutionis usum et applicationem mox diligentius evolvemus.

67. **Scholion 1.** Binae aequationes differentio-differentiales in alias transformari possunt ... anguli φ sinus et cosinus elidantur. Uti enim II. $\cos\varphi$ — I. $\sin\varphi$ praebet $2dvd\varphi + vdd\varphi = 0$ ... I. $\cos\varphi$ + II. $\sin\varphi$ suppeditat hanc aequationem

$$ddv - vd\varphi^2 = \frac{-2g(A+B)}{vv}dt^2,$$

quarum illa per v multiplicata et integrata statim praebet, ut vidimus, $vvd\varphi = Cdt$, qua aequa... arearum descriptio continetur. Deinde posterior per $2dv$, prior vero per $2vd\varphi$ multiplicata in ... summa praebent

$$2dvddv + 2vdvd\varphi^2 + 2vvd\varphi dd\varphi = \frac{-4g(A+B)dv}{vv}dt^2,$$

quae integrata dat:

$$dv^2 + vvd\varphi^2 = Ddt^2 + \frac{4g(A+B)}{v}dt^2,$$

ubi $\sqrt{(dv^2 + vvd\varphi^2)}$ exprimit elementum spatii Bb tempusculo dt descripti, inde autem ... $vvd\varphi^2 = \frac{CCdt^2}{vv}$ altera aequatio integralis ante inventa elicitur. Juvabit autem has aequationes ... ribus modis tractare, ut deinceps, cum hujusmodi aequationes magis complicatae occurrent, subsidi... inde peti queant. Licet etiam has duas aequationes

$$2dvd\varphi + vdd\varphi = 0 \quad \text{et} \quad ddv - vd\varphi^2 + \frac{2g(A+B)}{vv}dt^2 = 0$$

hoc modo resolvere: Multiplicetur prior per $2v^3d\varphi$, ut habeatur $4v^3dvd\varphi^2 + 2v^4d\varphi dd\varphi = 0$, cujus integrale est $v^4d\varphi^2 = EEdt^2$, unde valor pro dt^2 in altera aequatione substitutus praebet

$$ddv - vd\varphi^2 + \frac{2g(A+B)vvd\varphi^2}{EE} = 0.$$

Cum autem hic adhuc sit dt constans assumtum, ut ejus loco $d\varphi$ tanquam constans introducatur, multiplicetur per $2dv$, ut habeatur

$$2dvddv - 2vdvd\varphi^2 + \frac{4g(A+B)vvdv}{EE}d\varphi^2 = 0$$

et loco $2dvddv$ scribatur

$$dt^2d\cdot\frac{dv^2}{dt^2} = \frac{v^4d\varphi^2}{EE}d\cdot\frac{EEdv^2}{v^4d\varphi^2} = v^4d\cdot\frac{dv^2}{v^4}$$

et nunc elementum $d\varphi$ est constans. Statuatur porro $v = \frac{f}{u}$, erit $\frac{dv}{vv} = \frac{-du}{f}$ et $vdv = ...$ sicque prodibit

$$\frac{f^4}{u^4}d\cdot\frac{du^2}{ff} + \frac{2ffdud\varphi^2}{u^3} - \frac{4g(A+B)f^3du}{EEu^4}d\varphi^2 = 0,$$

seu
$$\frac{2duddu}{u^4} + \frac{2dud\varphi^2}{u^3} - \frac{4fg(A+B)du}{EEu^4}d\varphi^2 = 0,$$

vel
$$2duddu + 2udud\varphi^2 - \frac{4fg(A+B)du}{EE}d\varphi^2 = 0,$$

cujus integrale est

$$du^2 + uu\,d\varphi^2 - \frac{4fg(A+B)\,ud\varphi^2}{EE} = \frac{Dd\varphi^2}{EE}.$$

unde colligitur

$$d\varphi = \frac{Edu}{\sqrt{(D + 4fg(A+B)u - EEuu)}},$$

quae formula cum ante inventa congruit.

68. **Scholion 2.** Integralia formulae $\frac{ds}{(1+n\cos s)^2}$, quae prout fuerit $n<1$, vel $n>1$, vel $n=1$ exhibuimus, per se sunt manifesta, uti ex differentiatione patet. Est ergo casu $n<1$,

$$\int\frac{ds}{(1+n\cos s)^2} = \frac{1}{(1-nn)^{\frac{3}{2}}}\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} - \frac{n\sin s}{(1-nn)(1+n\cos s)};$$

sin autem sit $n>1$, erit

$$\int\frac{ds}{(1+n\cos s)^2} = \frac{n\sin s}{(nn-1)(1+n\cos s)} - \frac{1}{(nn-1)^{\frac{3}{2}}}\log.\frac{n+\cos s+\sin s\sqrt{(nn-1)}}{1+n\cos s},$$

quarum formularum utraque casu $n=1$ fit inepta; hoc autem casu $n=1$ habetur

$$\int\frac{ds}{(1+\cos s)^2} = \frac{(2+\cos s)\sin s}{3(1+\cos s)^2},$$

ubi praecipue notari meretur, quod in integrali eadem denominatoris potestas occurrit, atque in differentiali, cum alias sit unitate inferior. Simili modo est

$$\int\frac{ds}{1+\cos s} = \frac{\sin s}{1+\cos s}$$

atque adeo in genere formula $\int\frac{ds}{(1+\cos s)^n}$ integrari potest, cum sit

$$\int\frac{ds}{(1+\cos s)^n} = \frac{n-1}{2n-1}\int\frac{ds}{(1+\cos s)^{n-1}} + \frac{1}{2n-1}\cdot\frac{\sin s}{(1+\cos s)^n},$$

unde sequentia integralia deducuntur

$$\int\frac{ds}{1+\cos s} = \frac{\sin s}{1+\cos s},$$

$$\int\frac{ds}{(1+\cos s)^2} = \frac{\sin s(2+\cos s)}{3(1+\cos s)^2},$$

$$\int\frac{ds}{(1+\cos s)^3} = \frac{\sin s(7+6\cos s+2\cos^2 s)}{3.5(1+\cos s)^3},$$

$$\int\frac{ds}{(1+\cos s)^4} = \frac{\sin s(36+39\cos s+24\cos^2 s+6\cos^3 s)}{3.5.7(1+\cos s)^4},$$

$$\int\frac{ds}{(1+\cos s)^5} = \frac{\sin s(249+300\cos s+252\cos^2 s+120\cos^3 s+24\cos^4 s)}{3.5.7.9(1+\cos s)^5},$$

etc.

quae evanescunt posito $s=0$; ubi notandum si post integrationem ponatur $s=90^0$, fore

$$\int\frac{ds}{(1+\cos s)^n} = \frac{1}{2n-1} + \frac{n-1}{(2n-1)(2n-3)} + \frac{(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} + \frac{(n-1)(n-2)(n-3)}{(2n-1)(2n-3)(2n-5)(2n-7)}\text{ etc.},$$

quae series ad hanc progressionem infinitam reducitur

$$\frac{\sqrt{2}}{2n-1}+\frac{1.\sqrt{2}}{4(2n-3)}+\frac{1.3\sqrt{2}}{4.8(2n-5)}+\frac{1.3.5\sqrt{2}}{4.8.12(2n-7)}+\text{etc.}$$

Verum ad propositum revertentes, videamus hujusmodi curvam corpus B sit descripturum, et qua lege per eam sit progressurum, ita ut ad datum quodvis tempus locus corporis assignari possit, quod fit cum distantiam $AB=v$, tum angulum $EAB=\varphi$ pro tempore t definiendo.

69. **Problema.** Definire (fig. 176) naturam curvae EB, quam corpus B motu suo absolute spectato describit.

Solutio. Hic nullo respectu ad tempus habito, tantum ad relationem inter distantiam $AB=v$ et angulum $EAB=\varphi$ est spectandum, quae per angulum s ita definiuntur, ut sit $\varphi=s+\alpha$ et $v=\frac{f}{1+n\cos s}$. Statuantur coordinatae $AX=x$, $BX=y$, erit $vv=xx+yy$ et $x=v\cos\varphi$, $y=v\sin\varphi$, seu $\cos\varphi=\frac{x}{v}$; $\sin\varphi=\frac{y}{v}$. Cum ergo sit $s=\varphi-\alpha$, erit $\cos s=\frac{x\cos\alpha+y\sin\alpha}{v}$, unde fit $v+nx\cos\alpha+ny\sin\alpha=f$, ideoque

$$vv=xx+yy=(f-nx\cos\alpha-ny\sin\alpha)^2,$$

unde patet curvam esse sectionem conicam, cujus natura et positio ex aequatione $v=\frac{f}{1+n\cos(\varphi-\alpha)}$ facilius intelligitur. Ac primo quidem liquet, si sit $n=0$, ob $v=f$, curvam fore circulum centro A et radio $=f$ descriptum. Deinde si n sit numerus quicunque positivus, angulus $\varphi=\alpha$ dat minimam distantiam curvae a puncto A, quae est $=\frac{f}{1+n}$, ubi est $dv=0$ ob $dv=\frac{nf\,d\varphi\sin(\varphi-\alpha)}{(1+n\cos(\varphi-\alpha))^2}$. Simili modo sumendo $\varphi-\alpha=180^0$, prodit alter locus, ubi recta AB ad curvam est normalis, estque tum $v=\frac{f}{1-n}$, unde patet si sit $n<1$, curvam fore ellipsin; si $n=1$, parabolam; at si $n>1$, hyperbolam. Tum vero quia distantia $AB=v$ per coordinatas rationaliter exprimitur, punctum A in altero foco sectionis conicae est situm, cujus bini habentur vertices, quorum alterius a foco A distantia est $=\frac{f}{1+n}$, alterius $=\frac{f}{1-n}$, ita ut totus axis transversus sit $=\frac{2f}{1-nn}$, ideoque ejus semissis $=\frac{f}{1-nn}$, unde focus a centro sectionis distat intervallo $=\frac{nf}{1-nn}$; semiaxis ergo conjugatus erit $=\frac{f}{\sqrt{(1-nn)}}$, ideoque semiparameter $=f$. Axis denique transversus ad rectam fixam AE inclinatur angulo $=\alpha$, seu sumto $EAB=\alpha$, is in rectam AB cadet.

70. **Coroll. 1.** Curva ergo a corpore B circa A descripta semper est sectio conica, et cum sumto angulo $\varphi=\alpha$, corpus B transeat per verticem foco A propiorem, post singulas revolutiones completas, ubi $\varphi=\alpha+360^0$, $\varphi=\alpha+2.360^0$ etc. eodem revertitur, ita ut orbita haec quiescens sit censenda.

71. **Coroll. 2.** Cum valor numeri n speciem sectionis conicae ita determinet, ut $n=0$ det circulum, $n<1$ ellipsin, $n=1$ parabolam, et $n>1$ hyperbolam, idem intelligendum est, si n sit numerus negativus. In genere enim idem numerus n, sive sit positivus, sive negativus, eandem speciem declarat, quia scribendo $s+180^0$ loco s alter casus ad alterum reducitur.

72. **Scholion.** Praeter denominationes hic adhibitas notandae sunt sequentes ab Astronomis receptae:

Axis transversus sectionis conicae vocatur etiam *linea absidum*, ejusque terminus alter foco propior absis *ima*, alter remotior absis *summa*.

II. Distantia foci A a centro sectionis conicae per semiaxem transversum divisa, seu binorum focorum distantia per axem transversum ipsum divisa, vocatur *excentricitas* orbitae, quae ergo nostro calculo numero n exprimitur.

III. Angulus ad focum A, quem recta AB cum linea absidum facit, vocari solet *anomalia vera*; vulgo quidem hic angulus ad absidem summam refertur. Nihil autem impedit, quominus ad absidem imam referamus, quandoquidem corpus B, si orbita fuerit vel parabola vel hyperbola, nunquam ad absidem summam pervenit, semper autem per imam transit. (Fig. 177) Ita si C sit absis ima, corpusque ex C ad B pervenerit, angulum CAB vocabo anomaliam veram, quam ergo in calculo nostro littera s denotat.

IV. Angulus autem EAB a recta quadam fixa AE computatus vocari solet *longitudo*, quae hic nobis littera φ exprimitur. Simili modo angulus EAC est longitudo absidis imae C, unde patet longitudinem φ inveniri, si ad anomaliam veram s longitudo absidis imae EAC addatur.

His praemissis ipsam motus rationem, prout orbita fuerit vel circulus, vel ellipsis, vel parabola, vel hyperbola, investigemus.

73. **Problema.** Si orbita, in qua corpus B circa A revolvi videtur, fuerit circulus, definire rationem motus.

Solutio. Erit ergo excentricitas $n=0$, et radius circuli simulque perpetua distantia $AB=v=f$, unde si ponatur tempus, quo angulus $EAB=\varphi$ percurritur, $=t$, ob $ds=d\varphi$, habebitur $t=\frac{f\varphi\sqrt{f}}{\sqrt{2g(A+B)}}$; unde cum tempus sit ipsi angulo φ, ideoque et arcui $EB=f\varphi$ proportionale, motus erit uniformis, ejusque celeritas $=\frac{f\varphi}{t}=\sqrt{\frac{2g(A+B)}{f}}$, quae propterea est directe ut $\sqrt{(A+B)}$ et reciproce ut $\sqrt{f}$. Ac si tempus, quo totus circulus percurritur, quodque tempus *periodicum* vocatur, ponatur $=T$, ob $\varphi=2\pi$, erit

$$T=\frac{2\pi f\sqrt{f}}{\sqrt{2g(A+B)}}=\frac{2\pi}{\sqrt{2g}}\cdot\frac{f\sqrt{f}}{\sqrt{(A+B)}}.$$

Ergo ob $\frac{2\pi}{\sqrt{2g}}$ quantitatem constantem, tempus periodicum est directe in ratione sesquiplicata radii circuli f et reciproce in ratione subduplicata summae massarum $A+B$ utriusque corporis. Tempore ergo periodico cognito T, ob $\frac{t}{T}=\frac{\varphi}{2\pi}$, quovis tempore t percurritur arcus φ, ut sit $\varphi=\frac{2\pi t}{T}$, unde ad quodvis tempus t locus corporis B, ejus scilicet longitudo EAB facile colligitur. Pro mensura autem temporis absoluta definienda, consideretur ea corporum A et B distantia, in qua vis eorum attractrix aequalis est gravitati, quae distantia sit $=d$, eritque $\frac{A+B}{dd}=1$, seu $A+B=dd$; ubique ergo summa massarum $A+B$ per ejusmodi constantem multiplicari est censenda, ut fiat productum dd. Hinc ergo tempus t in minutis secundis exprimendo fiet $t=\frac{f\varphi\sqrt{f}}{d\sqrt{2g}}$, ideoque totum tempus periodicum $T=\frac{2\pi f\sqrt{f}}{d\sqrt{2g}}$ min. sec.

74. **Coroll. 1.** Ex dato ergo tempore periodico T pro quovis tempore dato t angulus ... ea descriptus φ per hanc analogiam facile definitur $T:t = 360^0:\varphi$; unde pro diebus, horis, ... et secundis motus angularis φ assignatur.

75. **Coroll. 2.** Ex tempore etiam periodico T et radio circuli descripti f, determinatur distantia corporum d, in qua eorum vis attractrix, qua B ad A urgetur, aequalis est ponderi corporis B, cum sit $d = \frac{2\pi f\sqrt{f}}{T\sqrt{2g}}$, ubi perpetuo est tenendum, tempora in minutis secundis exprimi oportere.

76. **Coroll. 3.** Si idem corpus B modo in majori modo in minori distantia circa corpus ... in circulo revolvatur, erunt tempora periodica in ratione sesquiplicata radiorum, seu quadrata temporum periodicorum erunt ut cubi radiorum.

77. **Problema.** Si orbita, in qua corpus B ex A spectatum moveri videtur, fuerit ellipsis, rationem motus definire.

Solutio. Erit ergo numerus n, quo excentricitas exprimitur, unitate minor, ac si ponatur semiparameter $= f$, anomalia vera seu angulus $CAB = s$, erit absidis imae C distantia $AC = \frac{f}{1+n}$, semiaxis transversus $= \frac{f}{1-nn}$, et semiaxis conjugatus $= \frac{f}{\sqrt{(1-nn)}}$. Statuatur autem longitudo absidis imae C seu angulus $EAC = \alpha$, a directione fixa AE computata, a qua corpus B egressum elapso tempore $= t$ pervenerit in B, ut sit longitudo ejus $EAB = \varphi$, erit $\varphi = \alpha + s$, ac habebimus has aequationes: Posita distantia $AB = v$:

$$v = \frac{f}{1+n\cos s} \quad \text{et} \quad t = \frac{f\sqrt{f}}{\sqrt{2g(A+B)}}\left(\frac{1}{(1-nn)^{\frac{3}{2}}}\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} - \frac{n\sin s}{(1-nn)(1+n\cos s)}\right),$$

quae formula proprie indicat tempus, quo corpus ab abside ima C in B usque pervenit, anomaliamque veram $CAB = s$ absolvit, quod tempus primo definiri convenit, cum deinceps ex eo tempus pro angulo $EAB = \varphi$ haud difficulter concludatur. Cum igitur posito $s = 0$ fiat $t = 0$, statuamus $s = 180^0$, erit tempus ab abside ima ad summam

$$= \frac{f\sqrt{f}}{\sqrt{2g(A+B)}} \cdot \frac{\pi}{(1-nn)^{\frac{3}{2}}},$$

cui iterum aequale fit tempus a summa ad imam, ita ut totum tempus periodicum, quod ponatur $= T$, futurum sit

$$= \frac{2\pi f\sqrt{f}}{(1-nn)^{\frac{3}{2}}\sqrt{2g(A+B)}},$$

quo tempore integra revolutio seu anomalia vera $= 360^0$ absolvitur. Hinc si motus angularis esset aequabilis tempore t, conficeretur angulus $= \tau$, ut sit

$$\frac{2\pi f\sqrt{f}}{(1-nn)^{\frac{3}{2}}\sqrt{2g(A+B)}} : t = 360^0 : \tau, \quad \text{seu} \quad t = \frac{\tau f\sqrt{f}}{(1-nn)^{\frac{3}{2}}\sqrt{2g(A+B)}};$$

ideoque ex cognito tempore periodico T, si motus esset aequabilis ad quodvis tempus elapsum postquam corpus absidem imam C fuerit transgressum, angulus interea confectus $\tau = \frac{360\,t}{T}$ assignari potest, quem ergo loco temporis t in calculum introducamus et inquiramus, quantum angulus interea confectus, seu anomalia vera $CAB = s$ ab eo discrepet. Pro t autem illo valore substituto habebimus

$$\frac{\tau f \sqrt{f}}{(1-nn)^{\frac{3}{2}}\sqrt{2g(A+B)}} = \frac{f\sqrt{f}}{\sqrt{2g(A+B)}}\left(\frac{1}{(1-nn)^{\frac{3}{2}}}\,\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} - \frac{n\sin s}{(1-nn)(1+n\cos s)}\right),$$

sive

$$\tau = \text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} - \frac{n\sin s\sqrt{(1-nn)}}{1+n\cos s}.$$

Ponatur $\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} = \sigma$, erit

$$\frac{n+\cos s}{1+n\cos s} = \cos\sigma \quad \text{et} \quad \sin\sigma = \frac{\sin s\sqrt{(1-nn)}}{1+n\cos s}, \quad \text{hincque}$$

$$\cos s = \frac{\cos\sigma - n}{1-n\cos\sigma} \quad \text{et} \quad \sin s = \frac{\sin\sigma\sqrt{(1-nn)}}{1-n\cos\sigma} \quad \text{atque} \quad \tau = \sigma - n\sin\sigma,$$

unde pro quovis angulo τ tempori t proportionali haud difficulter colligitur angulus σ, hincque porro anomalia vera s, cui si addatur longitudo absidis imae $EAC = \alpha$, obtinebitur longitudo quaesita seu angulus $EAB = \varphi$; distantia autem $AB = \rho$ ope formulae $\rho = \frac{f}{1+n\cos s}$ facillime assignatur.

78. **Coroll. 1.** Cum tempus periodicum sit

$$T = \frac{2\pi f\sqrt{f}}{(1-nn)^{\frac{3}{2}}\sqrt{2g(A+B)}}$$

semiaxis autem transversus orbitae $= \frac{f}{1-nn}$, qui si dicatur $= a$, erit tempus periodicum $T = \frac{2\pi a\sqrt{a}}{\sqrt{2g(A+B)}}$, quod ergo est directe in ratione sesquiplicata axis transversi, et reciproce in subduplicata summae massarum.

79. **Coroll. 2.** Simili modo si loco semiparametri f introducatur semiaxis transversus $a = \frac{f}{1-nn}$, habebitur pro tempore quocunque t

$$t = \frac{a\sqrt{a}}{\sqrt{2g(A+B)}}\left(\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} - \frac{n\sin s\sqrt{(1-nn)}}{1+n\cos s}\right),$$

$$\text{seu} \quad t = \frac{T}{2\pi}(\sigma - n\sin\sigma) \quad \text{posito} \quad \sigma = \text{Arc.}\cos\frac{n+\cos s}{1+n\cos s}.$$

80. **Coroll. 3.** Cognito ergo tempore periodico T et momento, quo corpus per absidem imam transiit, pro tempore inde elapso $= t$, quaeratur primo angulus $\tau = \frac{t}{T}\cdot 360^0$, hincque porro angulo σ, ut sit $\tau = \sigma - n\sin\sigma$, quo invento pro anomalia vera s habebitur

$$\cos s = \frac{\cos\sigma - n}{1-n\cos\sigma} \quad \text{seu} \quad \sin s = \frac{\sin\sigma\sqrt{(1-nn)}}{1-n\cos\sigma},$$

ac denique longitudo $\varphi = \alpha + s$.

81. **Scholion 1.** Hic iterum novae appellationes in Astronomia occurrunt quas probe notari convenit.

I. Angulus ille tempori proportionalis τ, qui pro revolutione integra abit in 360^0, voc anomalia media, quae ergo est angulus, quem corpus ab abside ima digressum, si aequabiliter punctum A eodem tempore periodico revolveretur, dato tempore esset confecturum.

II. Differentia inter anomaliam mediam τ et veram s vocari solet *aequatio centri*, vel *prostaphaeresis*, quae igitur est nulla casibus $\tau = 0$, $\tau = 180^0$, $\tau = 360^0$ etc., hoc est quo anomalia media in lineam absidum incidit.

III. Angulus ille subsidiarius σ, cujus relatio tam ad anomaliam mediam τ quam ad veram est assignata, vocari solet *anomalia excentrica*. Ex qua etiam distantia $AB = \varrho$ expedite defini Cum enim sit $1 + n\cos s = \frac{1-nn}{1-n\cos\sigma}$ et $\frac{f}{1-nn} = a$, erit $\varrho = \frac{f}{1+n\cos s} = a(1 - n\cos\sigma)$, ideoque distantia absidis imae a puncto $A = a(1-n)$, et summae $= a(1+n)$.

82. **Scholion 2.** Haec relatio inter anomalias veram, mediam et excentricam, quam per cal culum eruimus, ita geometrice doceri potest. Sit (fig. 178) AVB semiellipsis super axe transvers AB' descripta, cujus centrum in C et focus in F, positoque semiaxe $CA = a$ et excentricitate $=$ erit $CF = na$; tum super eodem axe constituatur semicirculus ANB. Sumta jam in ellipsi anom vera seu angulo $AFV = s$, ei respondeat in circulo anomalia media seu angulus $ACM = \tau$, at necesse est, ut sector circuli ACM sit ad aream semicirculi, ut sector ellipticus AFV ad aream sem ellipsis. Per V ducatur ad axem AB perpendicularis PVN circulum secans in N, ductaque recta est area elliptica AFV ad aream circularem AFN, ut semiellipsis area ad aream semicirculi, ex quo sectorem circularem ACM aequalem esse oportet areae circulari AFN. Unde notata rectarum FN et CM intersectione O, trilineum mixtilineum MON aequale esse debet triangulo rectilineo COF. Addatur utrinque triangulum CON ducto radio CN, ut fiat sector CMN aequalis triangulo CFN. Nunc primo patet angulum ACN esse anomaliam excentricam σ, nam hinc fit

$$PN = a\sin\sigma \quad \text{et} \quad PV = a\sin\sigma\sqrt{(1-nn)},$$

tum vero est $CP = a\cos\sigma$ et $FP = a(\cos\sigma - n)$, hincque $FV = a(1 - n\cos\sigma)$; unde fit invenimus

$$\cos s = \frac{\cos\sigma - n}{1 - n\cos\sigma}, \quad \text{seu} \quad \sin s = \frac{\sin\sigma\sqrt{(1-nn)}}{1-n\cos\sigma}.$$

Porro ob angulum $MCN = \sigma - \tau$, erit sector $MCN = \frac{1}{2}aa(\sigma - \tau)$, area vero trianguli $CFN = \frac{1}{2}naa\sin$ quibus valoribus aequatis fit $\sigma - \tau = n\sin\sigma$ seu $\tau = \sigma - n\sin\sigma$, quae aequalitas cum supra in venta congruit.

83. **Problema.** Data excentricitate orbitae ellipticae et anomalia media, invenire anomal excentricam, indeque anomaliam veram et aequationem centri seu prostaphaeresin.

Solutio. Posita excentricitate $= n$, et anomalia media $= \tau$, inde primo definiatur anomal excentrica σ ope aequationis $\tau = \sigma - n\sin\sigma$, quod commodissime per approximationem praesta Ponamus enim pro σ valorem jam prope verum esse inventum, qui sit $= \lambda$, et praebeat $\lambda - n\sin\lambda =$ ut error sit valde parvus δ, ac statuamus $\sigma = \lambda + \omega$, unde ob ω valde parvum, erit $\sin\sigma = \sin\lambda +$

ideoque $\tau = \lambda + \omega - n\sin\lambda - n\omega\cos\lambda = \tau + \delta + \omega - n\omega\cos\lambda$. Erit ergo $\omega = \frac{-\delta}{1-n\cos\lambda}$, ac propterea $\sigma = \lambda - \frac{\delta}{1-n\cos\lambda}$. Si valor λ non ita prope ad verum accedat, ut haec approximatio sufficiat, hinc saltem multo propior colligitur, qui loco λ positus multo exactius ad veritatem perducet. Ceterum si anomaliae mediae τ convenire reperta fuerit anomalia excentrica σ, anomaliae mediae tantillum majori $\tau + d\tau$ conveniet anomalia excentrica $\sigma + d\sigma$, ut sit $d\tau = d\sigma - nd\sigma\cos\sigma$, ideoque $d\sigma = \frac{d\tau}{1-n\cos\sigma}$, unde facile ad singulos gradus anomaliae mediae τ assignabitur anomalia excentrica σ. Inventa autem anomalia excentrica σ, anomalia vera s definiri debet ex hac formula

$$\cos s = \frac{\cos\sigma - n}{1-n\cos\sigma} \quad \text{seu} \quad \sin s = \frac{\sin\sigma\sqrt{(1-nn)}}{1-n\cos\sigma},$$

quae ut per logarithmos expediri posset, quaeratur primo angulus ω, ut sit $\tang\omega = \frac{\tang\sigma}{\sqrt{(1-nn)}}$, quo invento erit $\sin(s-\omega) = n\sin\omega$; seu quaeratur angulus ψ, ut sit $\sin\psi = n\sin\omega$, habebiturque $s = \omega + \psi$. Cum enim inde fiat $\sin s\cos\omega - \cos s\sin\omega = n\sin\omega$, erit

$$\tang\omega = \frac{\sin s}{n+\cos s} \quad \text{et} \quad \tang\sigma = \frac{\sin s\sqrt{(1-nn)}}{n+\cos s},$$

quae convenit cum formulis supra datis. Hinc denique erit aequatio centri $= s - \tau$ ad anomaliam mediam addenda, ut prodeat anomalia vera.

Ceterum notasse juvabit esse per formulas differentiales $ds = \frac{d\sigma\sqrt{(1-nn)}}{1-n\cos\sigma}$ et $d\sigma = \frac{ds\sqrt{(1-nn)}}{1+n\cos s}$. Quare cum sit $d\tau = d\sigma\,(1 - n\cos\sigma)$, erit

$$d\tau\, ds = d\sigma^2\sqrt{(1-nn)} = \frac{ds^2(1-nn)^{\frac{3}{2}}}{(1+n\cos s)^2}, \quad \text{ideoque} \quad d\tau = \frac{ds\,(1-nn)^{\frac{3}{2}}}{(1+n\cos s)^2}.$$

Unde si aequatio centri $s - \tau$ dicatur $= \varepsilon$, erit

$$d\varepsilon = ds - \frac{ds\,(1-nn)^{\frac{3}{2}}}{(1+n\cos s)^2}.$$

84. **Coroll. 1.** Si anomalia media τ evanescit, etiam fit anomalia excentrica $\sigma = 0$, unde quoque anomalia vera s et aequatio centri evanescit. Simili modo si anomalia media τ ponatur $= 180^0$, erit etiam $\sigma = 180^0$ et $s = 180^0$, ita ut etiam hoc casu aequatio centri evanescat.

85. **Coroll. 2.** Si anomalia media τ fuerit valde parva, erit etiam excentrica σ valde parva et $\sigma = \frac{\tau}{1-n}$, ob $\sin\sigma = \sigma$, hincque

$$ds = \frac{d\sigma\sqrt{(1-nn)}}{1-n} = \frac{d\tau\sqrt{(1-nn)}}{(1-n)^2}, \quad \text{unde} \quad s = \frac{\tau\sqrt{(1+n)}}{(1-n)\sqrt{(1-n)}}$$

et aequatio centri $s - \tau = \tau\left(-1 + \frac{\sqrt{(1+n)}}{(1-n)\sqrt{(1-n)}}\right)$, ideoque $s > \tau$.

86. **Coroll. 3.** Crescente anomalia media τ aequatio centri $s - \tau$ tamdiu crescit, quoad fiat

$$1 - \frac{(1-nn)^{\frac{3}{2}}}{(1+n\cos s)^2} = 0,$$

quo casu est maxima; tum iterum decrescit, donec posito $\tau = 180^0$ plane evanescat.

87. **Coroll. 4.** Aequatio centri ergo $s-\tau$ maxima evadit si

$$1+n\cos s=(1-nn)^{\frac{3}{4}} \quad \text{et} \quad \cos s=\frac{(1-nn)^{\frac{3}{4}}-1}{n},$$

unde fit

$$\cos\sigma=\frac{1-(1-nn)^{\frac{1}{4}}}{n} \quad \text{et} \quad \tau=\sigma-n\sin\sigma,$$

quae erit anomalia media, cui maxima aequatio centri convenit. Pro ea ergo erit $s>90^0$, $\sigma<90$ multoque magis $\tau<90$.

88. **Coroll. 5.** Sumta anomalia media τ negativa, fiunt quoque anomaliae σ et s negativae ejusdem valoris, unde binis anomaliis mediis τ et $360^0-\tau$ par respondet aequatio centri, quae autem priori casu est addenda, posteriori subtrahenda.

89. **Scholion.** Dum ergo corpus ab abside ima ad summam progreditur, aequatio centri est positiva, seu anomaliae mediae addenda, et quidem ab ima usque ad certum terminum continuo crescit, unde ad absidem summam usque iterum decrescit, ubi evanescit. Tum vero ab abside summa ad imam progrediendo per pares aequationes anomalia media est minuenda, unde sufficit aequationes centri nosse pro transitu ab abside ima ad summam. Si enim anomaliae mediae τ conveniat aequatio centri ε, anomaliae mediae $360^0-\tau$ conveniet aequatio centri $-\varepsilon$. Modus autem hic expositus ex data anomalia media computandi anomaliam veram commodior reddi potest, si excentricitas fuerit valde parva, id quod plerumque usu venit, unde hunc casum seorsim evolvisse juvabit.

90. **Problema.** Si excentricitas n fuerit valde parva, pro data anomalia media definire aequationem centri et anomaliam veram.

Solutio. Primo ex anomalia media τ colligitur anomalia excentrica σ ope aequationis $\tau=\sigma-n\sin\sigma$, unde erit

$$\sigma=\tau+n\sin(\tau+n\sin(\tau+n\sin(\tau+n\sin(\tau+\text{etc.},$$

vel etiam ope hujus formulae

$$\sigma=\tau+(n-\tfrac{1}{8}n^3)\sin\tau+(\tfrac{1}{2}nn-\tfrac{1}{6}n^4)\sin 2\tau+\tfrac{3}{8}n^3\sin 3\tau+\tfrac{1}{3}n^4\sin 4\tau,$$

ubi potestates ipsius n quarta altiores sunt neglectae. Inventa autem anomalia excentrica σ, ex anomalia vera s definitur hac aequatione $ds=\frac{d\sigma\sqrt{(1-nn)}}{1-n\cos\sigma}$, unde fit

$$ds=d\sigma(1+n\cos\sigma+n^2\cos^2\sigma+n^3\cos^3\sigma+n^4\cos^4\sigma+n^5\cos^5\sigma+\text{etc.})\sqrt{(1-nn)}.$$

Cum igitur sit

$$\begin{aligned}
\cos\sigma &= \cos\sigma\\
\cos^2\sigma &= \tfrac{1}{2}+\tfrac{1}{2}\cos 2\sigma\\
\cos^3\sigma &= \tfrac{3}{4}\cos\sigma+\tfrac{1}{4}\cos 3\sigma\\
\cos^4\sigma &= \tfrac{3}{8}+\tfrac{4}{8}\cos 2\sigma+\tfrac{1}{8}\cos 4\sigma\\
\cos^5\sigma &= \tfrac{10}{16}\cos\sigma+\tfrac{5}{16}\cos 3\sigma+\tfrac{1}{16}\cos 5\sigma\\
&\text{etc.}
\end{aligned}$$

colligendis his terminis:

$$ds = (1 - nn)^{\frac{1}{2}} d\sigma \begin{cases} + (1 + \frac{1}{2} nn + \frac{3}{8} n^4 + \text{etc.}) \\ + (n + \frac{3}{4} n^3 + \frac{10}{16} n^5 + \text{etc.}) \cos\sigma \\ + (\frac{1}{2} nn + \frac{4}{8} n^4 + \text{etc.}) \cos 2\sigma \\ + (\frac{1}{4} n^3 + \frac{5}{16} n^5 + \text{etc.}) \cos 3\sigma \\ + (\frac{1}{8} n^4 + \text{etc.}) \cos 4\sigma \\ + \text{etc.} \end{cases}$$

Est vero

$$1 + \tfrac{1}{2} nn + \tfrac{3}{8} n^4 + \text{etc.} = (1 - nn)^{-\frac{1}{2}}$$

$$\text{et} \quad n + \tfrac{3}{4} n^3 + \tfrac{10}{16} n^5 + \text{etc.} = \frac{2}{n}\left((1 - nn)^{-\frac{1}{2}} - 1\right),$$

ita ut pro primis duobus terminis habeatur

$$ds = d\sigma\left(1 + \frac{2}{n}(1 - \sqrt{(1 - nn)}) \cos\sigma\right), \quad \text{ideoque} \quad s = \sigma + \frac{2}{n}(1 - \sqrt{(1 - nn)}) \sin\sigma.$$

Quo autem hanc seriem ulterius continuare queamus, ponamus

$$ds = d\sigma\,(1 + A\cos\sigma + B\cos 2\sigma + C\cos 3\sigma + D\cos 4\sigma + E\cos 5\sigma + \text{etc.})$$

et cum sit $\frac{ds}{d\sigma} - \frac{n\,ds}{d\sigma}\cos\sigma = \sqrt{(1 - nn)}$, ob $\cos\sigma\cos\nu\sigma = \frac{1}{2}\cos(\nu - 1)\,\sigma + \frac{1}{2}\cos(\nu + 1)\,\sigma$ fiet

$$\left.\begin{array}{llllll} 1 & + A\cos\sigma & + B\cos 2\sigma & + C\cos 3\sigma & + D\cos 4\sigma & + E\cos 5\sigma + \text{etc.} \\ -\frac{1}{2} nA & - n & -\frac{1}{2} nA & -\frac{1}{2} nB & -\frac{1}{2} nC & -\frac{1}{2} nD \\ & -\frac{1}{2} nB & -\frac{1}{2} nC & -\frac{1}{2} nD & -\frac{1}{2} nE & -\frac{1}{2} nF \end{array}\right\} = \sqrt{(1 - nn)}$$

unde coefficientes sequenti modo determinantur

$$A = \frac{2}{n}(1 - \sqrt{(1 - nn)}) \quad \text{seu} \quad A = 2\left(\frac{1 - \sqrt{(1 - nn)}}{n}\right),$$

$$B = \frac{2}{n}(A - n) \quad \text{seu} \quad B = 2\left(\frac{1 - \sqrt{(1 - nn)}}{n}\right)^2,$$

$$C = \frac{1}{n}(2B - nA) \quad \text{seu} \quad C = 2\left(\frac{1 - \sqrt{(1 - nn)}}{n}\right)^3,$$

$$D = \frac{1}{n}(2C - nB) \quad \text{seu} \quad D = 2\left(\frac{1 - \sqrt{(1 - nn)}}{n}\right)^4,$$

$$E = \frac{1}{n}(2D - nC) \quad \text{seu} \quad E = 2\left(\frac{1 - \sqrt{(1 - nn)}}{n}\right)^5,$$

$$F = \frac{1}{n}(2E - nD) \quad \text{seu} \quad F = 2\left(\frac{1 - \sqrt{(1 - nn)}}{n}\right)^6,$$

etc.

Ponatur brevitatis gratia $\frac{1-\sqrt{(1-nn)}}{n}=m$, erit

$$ds = d\sigma\,(1 + 2m\cos\sigma + 2m^2\cos 2\sigma + 2m^3\cos 3\sigma + 2m^4\cos 4\sigma + \text{etc.})$$

hincque integrando

$$s = \sigma + \tfrac{2}{1}m\sin\sigma + \tfrac{2}{2}m^2\sin 2\sigma + \tfrac{2}{3}m^3\sin 3\sigma + \tfrac{2}{4}m^4\sin 4\sigma + \text{etc.}$$

Cum nunc sit $\sigma = \tau + n\sin\sigma$, erit aequatio centri

$$s - \tau = (2m + n)\sin\sigma + \tfrac{2}{2}m^2\sin 2\sigma + \tfrac{2}{3}m^3\sin 3\sigma + \tfrac{2}{4}m^4\sin 4\sigma + \text{etc.}$$

Potest etiam anomalia media τ per veram s simili modo exprimi; cum enim sit

$$d\tau = \frac{ds\,(1-nn)^{\frac{3}{2}}}{(1+n\cos s)^2},$$

erit $$d\tau = (1-nn)^{\frac{3}{2}}\,ds\,(1 - 2n\cos s + 3n^2\cos^2 s - 4n^3\cos^3 s + 5n^4\cos^4 s - \text{etc.}),$$

cujus seriei, si potestates cosinus s ad cosinus multiplorum angulorum revocentur, prodit terminus constans

$$1 + \tfrac{3}{2}n^2 + \frac{3.5}{2.4}n^4 + \text{etc.} = (1-nn)^{-\frac{3}{2}}$$

et coefficiens ipsius $\cos s$ fit

$$= -2n\,(1-nn)^{-\frac{3}{2}}.$$

Quare ponamus

$$d\tau = ds\,(1 - A\cos s + B\cos 2s - C\cos 3s + D\cos 4s - \text{etc.}),$$

quae series per $(1+n\cos s)^2 = 1 + \frac{1}{2}nn + 2n\cos s + \frac{1}{2}nn\cos 2s$ multiplicata dat

$$\begin{array}{llllll}
1 + \frac{1}{2}nn & -A(1+\frac{1}{2}nn)\cos s & +B(1+\frac{1}{2}nn)\cos 2s & -C(1+\frac{1}{2}nn)\cos 3s & +D(1+\frac{1}{2}nn)\cos 4s & \text{etc.}\\
-An & +2n & -An & +Bn & -Cn & \\
+\frac{1}{4}Bnn & +Bn & -Cn & +Dn & -En & \\
 & -\frac{1}{4}Ann & +\frac{1}{2}nn & -\frac{1}{4}Ann & +\frac{1}{4}Bnn & \\
 & -\frac{1}{4}Cnn & +\frac{1}{4}Dnn & -\frac{1}{4}Enn & +\frac{1}{4}Fnn &
\end{array}$$

aequari debet ipsi $(1-nn)^{\frac{3}{2}}$. At est $A = 2n$, unde fit primo

$$1 + \tfrac{1}{2}nn - 2nn + \tfrac{1}{4}Bnn = (1-nn)^{\frac{3}{2}},$$

ideoque

$$B = \frac{4(1-nn)^{\frac{3}{2}} - 4 + 6nn}{nn} = \tfrac{3}{2}nn + \tfrac{1}{4}n^4,$$

$$C = \frac{4Bn - 4A - 3Ann + 8n}{nn} = \frac{16(1-nn)^{\frac{3}{2}} - 16 + 24nn - 6n^4}{n^3},$$

$$D = \frac{4Cn - 4B - 2Bnn + 4An - 2nn}{nn} = \frac{8(6-nn)(1-nn)^{\frac{3}{2}} - 48 + 80nn - 30n^4}{n^4},$$

$$E = \frac{4Dn - 4C - 2Cnn + 4Bn - Ann}{nn},$$

$$F = \frac{4En - 4D - 2Dnn + 4Cn - Bnn}{nn}.$$

Alio autem modo reperitur

$$A = 2n; \quad B = 2(1 + 2\sqrt{(1-nn)})\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^2; \quad C = \frac{4B - 3An}{n}; \quad D = \frac{6C - 4Bn}{2n};$$

$$E = \frac{8D - 5Cn}{3n}; \quad F = \frac{10E - 6Dn}{4n}.$$

Hincque coefficientes quaesiti sequenti modo exprimi invenirentur:

$$A = 2(1 + \sqrt{(1-nn)})\left(\frac{1-\sqrt{(1-nn)}}{n}\right),$$

$$B = 2(1 + 2\sqrt{(1-nn)})\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^2,$$

$$C = 2(1 + 3\sqrt{(1-nn)})\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^3,$$

$$D = 2(1 + 4\sqrt{(1-nn)})\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^4,$$

etc.

Quibus valoribus inventis erit

$$\tau = s - A\sin s + \tfrac{1}{2}B\sin 2s - \tfrac{1}{3}C\sin 3s + \tfrac{1}{4}D\sin 4s - \tfrac{1}{5}E\sin 5s + \text{etc.},$$

ita ut aequatio centri sit futura

$$s - \tau = A\sin s - \tfrac{1}{2}B\sin 2s + \tfrac{1}{3}C\sin 3s - \tfrac{1}{4}D\sin 4s + \tfrac{1}{5}E\sin 5s - \text{etc.}$$

91. **Coroll. 1.** Si n tam sit parvum, ut potestates omnes rejicere liceat, erit $\sigma = \tau + n\sin\tau$, ubi ob $m = \frac{1}{2}n$, fit $s = \sigma + n\sin\sigma = \tau + 2n\sin\tau$, ideoque aequatio centri $s - \tau = 2n\sin\tau$; quae ergo est maxima $= 2n$, sumta anomalia media $\tau = 90^0$.

92. **Coroll. 2.** Si tantum potestates secunda superiores rejicere liceat, ob $\frac{1-\sqrt{(1-nn)}}{n} = \frac{1}{2}n$, erit $A = 2n$, $B = \frac{3}{2}nn$, $C = 0$ etc., unde fit $\tau = s - 2n\sin s + \frac{3}{4}nn\sin 2s$; hincque per conver-

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sionem reperitur $s = \tau + 2n\sin\tau + \frac{5}{4}nn\sin 2\tau$, seu aequatio centri $s - \tau = 2n\sin\tau + \frac{5}{4}nn\sin 2\tau$, quae ergo est maxima si $\tau = \frac{\pi}{2} - \frac{5}{4}n$, fitque $= 2n$.

93. **Coroll. 3.** Si potestates ipsius n quarta altiores tantum rejicere liceat, erit

$$A = 2n - \frac{1}{2}n^3, \quad B = \frac{3}{2}nn - \frac{1}{2}n^4, \quad C = n^3, \quad D = \frac{5}{8}n^4, \quad E = 0, \quad \text{etc.}$$

ideoque habebitur

$$\tau = s - (2n - \tfrac{1}{2}n^3)\sin s + (\tfrac{3}{4}nn - \tfrac{1}{4}n^4)\sin 2s - \tfrac{1}{3}n^3\sin 3s + \tfrac{5}{32}n^4\sin 4s,$$

unde per conversionem eruitur

$$s = \tau + (2n - \tfrac{3}{4}n^3)\sin\tau + (\tfrac{5}{4}nn - \tfrac{13}{12}n^4)\sin 2\tau + \tfrac{13}{12}n^3\sin 3\tau + \tfrac{103}{96}n^4\sin 4\tau.$$

94. **Coroll. 4.** Aequatio centri ergo fit maxima ubi est

$$(2n - \tfrac{3}{4}n^3)\cos\tau + (\tfrac{5}{2}nn - \tfrac{13}{6}n^4)\cos 2\tau + \tfrac{13}{4}n^3\cos 3\tau + \tfrac{103}{24}n^4\cos 4\tau = 0,$$

unde colligitur

$$\tau = \frac{\pi}{2} - \frac{5}{4}n - \frac{25}{384}n^3,$$

ita ut haec anomalia media minor sit angulo recto. Ipsa autem aequatio maxima ex formula generali supra data facilius eruetur.

95. **Scholion.** Scilicet cum ex § 87 pro aequatione maxima sit proxime

$$\cos\sigma = \frac{1 - (1 - nn)^{\frac{1}{4}}}{n} = \frac{1}{4}n + \frac{3}{32}n^3,$$

$$\text{erit} \quad \sigma = \frac{\pi}{2} - \frac{1}{4}n - \frac{37}{384}n^3 \quad \text{et} \quad \sin\sigma = 1 - \frac{1}{32}nn - \frac{49}{2048}n^4.$$

Deinde vero est

$$\cos s = \frac{(1 - nn)^{\frac{3}{4}} - 1}{n} = -\frac{3}{4}n - \frac{3}{32}n^3,$$

$$\text{unde} \quad s = \frac{\pi}{2} + \frac{3}{4}n + \frac{21}{128}n^3 \quad \text{et} \quad \tau = \frac{\pi}{2} - \frac{5}{4}n - \frac{25}{384}n^3,$$

ideoque aequatio maxima $\qquad s - \tau = 2n + \frac{11}{48}n^3.$

Ceterum methodus priori loco exposita, qua primo anomaliam excentricam σ investigavimus, commodius adhiberi videtur, cum ejus ope appropinquatio facile longius extendi queat, quandoquidem serie qua s per σ exprimitur, lex progressionis est manifesta; ac si accuratius σ per τ exprimere velimus reperiemus

$$\sigma = \tau + (n - \tfrac{1}{8}n^3 + \tfrac{1}{192}n^5)\sin\tau + (\tfrac{1}{2}n^2 - \tfrac{1}{6}n^4)\sin 2\tau + (\tfrac{3}{8}n^3 - \tfrac{41}{192}n^5)\sin 3\tau + \tfrac{1}{3}n^4\sin 4\tau + \tfrac{21}{64}n^5\sin 5\tau$$

ubi tamen legem progressionis perspicere non licet.

96. **Problema.** Si curva, in qua corpus B circa A moveri cernitur, fuerit parabola, ad datum tempus assignare locum, ubi corpus B versabitur.

Solutio. Denotante f semiparametrum parabolae, ut distantia absidis imae C a foco A sit $AC = f$; si tempore t corpus ex C in B usque progrediatur, confecta anomalia vera $CAB = s$, erit distantia $AB = v = \frac{f}{1+\cos s}$, et si A et B corporum massas denotent, invenimus

$$t = \frac{f\sqrt{f}}{\sqrt{2g(A+B)}}\int\frac{ds}{(1+\cos s)^2} = \frac{f\sqrt{f}}{\sqrt{2g(A+B)}}\cdot\frac{(2+\cos s)\sin s}{3(1+\cos s)^2}.$$

Quo temporis rationem facilius tenere possimus, consideremus casum, quo corpus E circa aliud corpus F circulum uniformiter describit, cujus radius $= e$, atque tempore eodem t angulus descriptus sit $= \tau$, qui loco ipsius temporis t introducatur. Cum igitur sit $t = \frac{e\sqrt{e}}{\sqrt{2g(E+F)}}\cdot\tau$, statuamus brevitatis gratia $\frac{e\sqrt{e}}{f\sqrt{f}}\cdot\sqrt{\frac{A+B}{E+F}} = m$, ut obtineamus $m\tau = \frac{(2+\cos s)\sin s}{3(1+\cos s)^2}$, unde ex dato angulo τ definiri oportet angulum s, id quod resolutionem aequationis cubicae postulat. Praestabit autem tabulam computare, quae ad singulos gradus anguli s exhibeat valorem formulae $\frac{(2+\cos s)\sin s}{3(1+\cos s)^2}$, ex qua deinceps facile erit pro dato $m\tau$ respondentem anomaliam veram s colligere. Ad hunc calculum sublevandum notetur esse

$$m\tau = \frac{\left(1+2\cos^2\frac{1}{2}s\right)\sin\frac{1}{2}s}{6\cos^3\frac{1}{2}s} = \frac{\sin\frac{1}{2}s}{3\cos\frac{1}{2}s} + \frac{\sin\frac{1}{2}s}{6\cos^3\frac{1}{2}s},$$

seu $m\tau = \frac{1}{3}\tang\frac{1}{2}s + \frac{1}{6}\tang\frac{1}{2}s.\sec^2\frac{1}{2}s = \frac{1}{6}\tang\frac{1}{2}s\left(2+\sec^2\frac{1}{2}s\right) = \frac{1}{2}\tang\frac{1}{2}s + \frac{1}{6}\tang^3\frac{1}{2}s.$

Verumtamen etiam ex dato $m\tau$, modo hic angulus in partibus radii exponatur, angulus s definiri potest, nam ponatur $\tang\frac{1}{2}s = z$, erit $\sec^2\frac{1}{2}s = 1+zz$, unde fit $6m\tau = z(3+zz)$, ex cujus aequationis resolutione, si ponamus $3m\tau = u$, deducimus

$$z = \tang\tfrac{1}{2}s = \left(\sqrt{(1+uu)}+u\right)^{\frac{1}{3}} - \left(\sqrt{(1+uu)}-u\right)^{\frac{1}{3}},$$

vel quaeratur angulus ω, ut sit $\tang 2\omega = 3m\tau$, tum vero erit

$$\tang\tfrac{1}{2}s = \sqrt[3]{\tang(45^0+\omega)} - \sqrt[3]{\tang(45^0-\omega)},$$

sive $$\tang\tfrac{1}{2}s = \sqrt[3]{\tang(45^0+\omega)} - \sqrt[3]{\cot.(45^0+\omega)},$$

cujus ope calculus per logarithmos haud difficulter instituitur, quoniam

$$\log.\sqrt[3]{\tang(45^0-\omega)} = -\log.\sqrt[3]{\tang(45^0+\omega)}.$$

Fortasse calculus adhuc facilior reddetur, si quaeratur angulus ψ, ut sit

$$\operatorname{tang}(45^0 + \psi) = \sqrt[3]{\operatorname{tang}(45^0 + \omega)},$$

eritque $$\operatorname{tang}\tfrac{1}{2}s = 2\operatorname{tang}2\psi,$$

qui calculus ad logarithmorum usum maxime est accommodatus.

97. **Coroll. 1.** Elapso ergo tempore t post transitum corporis B per absidem imam, ipsum pro eodem tempore angulus τ definiatur, quem corpus quoddam E circa aliud F circulum radii $= e$ describens interea conficit, qui angulus loco temporis in calculum introducatur, tum vero capiatur numerus $m = \frac{e\sqrt{e}}{f\sqrt{f}}\sqrt{\frac{A+B}{E+F}}$, et factum $m\tau$ in partibus radii exprimatur eritque

$$m\tau = \tfrac{1}{2}\operatorname{tang}\tfrac{1}{2}s + \tfrac{1}{6}\operatorname{tang}^3\tfrac{1}{2}s.$$

98. **Coroll. 2.** Uti sumto $\tau = 0$, anomalia vera s evanescit, ita tempus usque ad absidem summam fit infinitum; sumta enim anomalia vera $s = 180^0$, ob $\operatorname{tang}\frac{1}{2}s = \infty$, evidens est quantitatem $m\tau$ in infinitum augeri, scilicet corpus in parabola motum nunquam ad absidem summam pertingit.

99. **Coroll. 3.** Si anomalia vera s fuerit valde parva, ob $\operatorname{tang}\frac{1}{2}s = \frac{1}{2}s + \frac{1}{24}s^3$ et $\operatorname{tang}^3\frac{1}{2}s = \frac{1}{8}s^3$, fiet $m\tau = \frac{1}{4}s + \frac{1}{24}s^3$. Sumta autem $s = 60^0$ ob $\operatorname{tang}30^0 = \frac{1}{\sqrt{3}}$ fit $m\tau = \frac{5}{9\sqrt{3}}$, at sumta $s = 90^0$, ob $\operatorname{tang}45^0 = 1$, fit $m\tau = \frac{2}{3}$. Sumta denique $s = 120^0$, ob $\operatorname{tang}60^0 = \sqrt{3}$ fit $m\tau = \sqrt{3}$.

100. **Coroll. 4.** Si ex dato tempore t seu angulo ipsi proportionali $m\tau$ anomaliam veram definire velimus, promptissime id praestabitur hoc modo: Primo quaeratur angulus ω, ut sit

$$\operatorname{tang}2\omega = 3m\tau,$$

tum angulus ψ ut sit

$$\operatorname{tang}(45^0 + \psi) = \sqrt[3]{\operatorname{tang}(45^0 + \omega)},$$

quo invento erit $\operatorname{tang}\frac{1}{2}s = 2\operatorname{tang}2\psi$.

101. **Scholion.** Quemadmodum hic aequationis cubicae $z^3 + 3z = 6m\tau$ resolutionem commode per tabulas sinuum docuimus, qui modus alias tantum in iis aequationibus cubicis usurpari solet, quae omnes radices habent reales, ita in genere aequationum unica radice reali praeditarum resolutio quoque ad tabulas sinuum revocari potest hoc modo: Sit sublato secundo termino proposita haec aequatio cubica $y^3 + 3by = 2c$, quaeratur angulus ω, ut sit

$$\operatorname{tang}2\omega = \frac{c}{b\sqrt{b}},$$

tum vero angulus ψ, ut sit

$$\operatorname{tang}(45^0 + \psi) = \sqrt[3]{\operatorname{tang}(45^0 + \omega)},$$

radix realis $y = 2\sqrt{b}\,\mathrm{tang}\,2\psi$. Hoc scilicet modo regula Cardani commodius ad calculum accommodatur. Vel etiam ex angulo ω statim est

$$y = (\sqrt[3]{\mathrm{tang}\,(45^0 + \omega)} - \sqrt[3]{\mathrm{tang}\,(45^0 - \omega)})\sqrt{b}.$$

102. **Problema.** Si curva, quam corpus B circa A describit, proxime tantum ad parabolam accedat, ad quodvis tempus ab abside ima elapsum locum corporis in curva assignare.

Solutio. Excentricitas ergo n unitati proxime aequalis assumitur, unde si ut ante tempus t ex motu uniformi, quo corpus quodpiam E circa aliud F ad distantiam $= e$ circulum describit, definiamus, atque in hoc circulo tempore t absolvatur angulus $= \tau$, ponaturque

$$\frac{e\sqrt{e}}{f\sqrt{f}}\sqrt{\frac{A+B}{E+F}} = m;$$

ex superioribus habemus elapso tempore t ab abside ima C distantiam

$$AB = v = \frac{f}{1 + n\cos s} \quad \text{et} \quad m\tau = \frac{1}{(1-nn)^{\frac{3}{2}}}\,\mathrm{Arc.cos}\,\frac{n+\cos s}{1+n\cos s} - \frac{n\sin s}{(1-nn)(1+n\cos s)}$$

denotante f semiparametrum orbitae et s anomaliam veram seu angulum CAB. Haec quidem formula pro casu, quo $n < 1$ et orbita est ellipsis, valet, unde patet pro tempore motus ab abside ima ad summam prodire

$$m\tau = \frac{\pi}{(1-nn)^{\frac{3}{2}}},$$

id quod ex approximationibus minus liquet, quippe quae non ad absidem summam usque extenduntur. Nam cum sit

$$\mathrm{Arc.cos}\,\frac{n+\cos s}{1+n\cos s} = \mathrm{Arc.sin}\,\frac{\sin s\sqrt{(1-nn)}}{1+n\cos s},$$

hic sinus quidem est valde parvus, quamdiu anomalia vera s non proxime ad 180^0 accedit. Cum igitur existente sinu u minimo sit

$$\mathrm{Arc.sin}\,u = u + \tfrac{1}{6}u^3 + \tfrac{3}{40}u^5 + \tfrac{5}{112}u^7 \text{ etc.}$$

erit $$\mathrm{Arc.sin}\,\frac{\sin s\sqrt{(1-nn)}}{1+n\cos s} = \frac{\sin s\sqrt{(1-nn)}}{1+n\cos s} + \frac{(1-nn)^{\frac{3}{2}}\sin^3 s}{6(1+n\cos s)^3} + \frac{3(1-nn)^{\frac{5}{2}}\sin^5 s}{40(1+n\cos s)^5},$$

ideoque nostra aequatio fiet

$$m\tau = \frac{\sin s}{(1+n)(1+n\cos s)} + \frac{\sin^3 s}{6(1+n\cos s)^3} + \frac{3(1-nn)\sin^5 s}{40(1+n\cos s)^5}.$$

Quoniam igitur n proxime ad unitatem accedit, sive sit $n < 1$ sive $n > 1$, formula inventa aeque locum habet, neque tamen eousque progredi licet, ut denominator $1 + n\cos s$ proxime evanescat. Ponamus ergo $n = 1 - \delta$ et $\mathrm{tang}\,\frac{1}{2}s = z$, ac reperiemus

$$m\tau = \frac{2z}{(2-\delta)(2-\delta+\delta zz)} + \frac{4z^3}{3(2-\delta+\delta zz)^3} + \frac{24\delta z^5}{5(2-\delta+\delta zz)^5}$$

neglectis terminis ubi δ plus una dimensione adipiscitur. Pro data ergo anomalia vera s, tempus ejusve loco $m\tau$ facillime colligetur hoc modo: Statuatur $\frac{\sin s}{1+n\cos s}=u$, eritque

$$m\tau=\frac{u}{1+n}+\tfrac{1}{6}u^3+\tfrac{3}{40}(1-nn)u^5.$$

At si discrimen a particula δ oriundum noscere velimus, ex $z=\operatorname{tang}\frac{1}{2}s$ obtinebimus

$$m\tau=\tfrac{1}{2}z+\tfrac{1}{6}z^3+\delta\left(\tfrac{1}{2}z-\tfrac{1}{10}z^5\right),$$

unde si neglecta particula δ pro dato $m\tau$ invenerimus $z=q$, erit ratione ipsius δ habita

$$z=q-\frac{\delta q(5-q^4)}{5(1+qq)}=\operatorname{tang}\tfrac{1}{2}s.$$

103. **Coroll. 1.** Si ergo particula minima δ fuerit positiva, curva erit ellipsis perquam longa cujus semiaxis transversus $=\frac{f}{1-nn}=\frac{f}{2\delta}$, et semiaxis conjugatus $=\frac{f}{\sqrt{2\delta}}$, atque distantia absidis imae a foco $=\frac{f}{2-\delta}=\frac{1}{2}f+\frac{\delta}{4}f$.

104. **Coroll. 2.** Sin autem particula δ fuerit negativa, curva erit hyperbola minime a parabola ejusdem parametri discrepans, cujus asymtotae ad axem erunt inclinatae angulo cujus cosinus $=\frac{1}{1+\delta}$ vel sinus $=\sqrt{2\delta}$.

105. **Coroll. 3.** Ceterum calculus, quo tam ex data anomalia vera s quaeritur quantitas tempori proportionalis $m\tau$, quam vicissim haec ex illa, non multo onerosior est illo, quem ante pro parabola docuimus, unde ad motum in hyperbola scrutandum procedamus.

106. **Problema.** Si curva, in qua corpus B circa A moveri videtur, fuerit hyperbola, ad quodvis tempus ejus locum assignare.

Solutio. Loco temporis t introducamus et hic quantitatem ipsi proportionalem $m\tau$ modo ante expositam, et cum numerus n excentricitatem referens sit unitate major, ex § 68 habebimus

$$m\tau=\frac{n\sin s}{(nn-1)(1+n\cos s)}-\frac{1}{(nn-1)^{\frac{3}{2}}}\log.\frac{n+\cos s+\sin s\sqrt{(nn-1)}}{1+n\cos s},$$

qua aequatione relatio inter tempus et anomaliam veram $CAB=s$ exprimitur. Hic autem logarithmus ex canone logarithmorum hyperbolicorum sumi debet, vel si logarithmum vulgarem capiamus eum per numerum 2.30258509299 multiplicari, hujusve reciprocum 0,4342944819 dividi oportet. Statuamus $\frac{n+\cos s}{1+n\cos s}=u$, ut sit $\cos s=\frac{n-u}{nu-1}$, et quia $\sqrt{(uu-1)}=\frac{\sin s\sqrt{(nn-1)}}{1+n\cos s}$, nostra aequatio erit

$$m\tau=\frac{n}{(nn-1)^{\frac{3}{2}}}\sqrt{(uu-1)}-\frac{1}{(nn-1)^{\frac{3}{2}}}\log.(u+\sqrt{(uu-1)}),$$

quae aequatio adhuc simplicior reddi potest ponendo $u=\sec 2\omega=\frac{1}{\cos 2\omega}$, seu $\sqrt{(uu-1)}=\operatorname{tang}2\omega$

[illegible] $\frac{\ldots n\cos s}{n+\cos s} = \cos 2\omega$, hincque $\cos s = \frac{n\cos 2\omega - 1}{n - \cos 2\omega}$ et $\sin s = \frac{\sin 2\omega \sqrt{(nn-1)}}{n - \cos 2\omega}$; tum enim erit [illegible] $\sqrt{(nn-1)}) = \operatorname{tang}(45^0 + \omega)$, ideoque

$$m\tau = \frac{n \operatorname{tang} 2\omega - \log . \operatorname{tang}(45^0 + \omega)}{(nn-1)\sqrt{(nn-1)}}.$$

Quodsi ergo ad singulos gradus anomaliae verae s valores quantitatis $m\tau$ computentur, inde vicissim pro dato $m\tau$ ipsa anomalia vera s simulque distantia $AB = \varrho = \frac{f}{1 + n\cos s}$ facile colligitur.

107. **Coroll. 1.** Crescente ergo tempore t seu quantitate ipsi proportionali $m\tau$, crescit etiam anomalia vera $CAB = s$, atque elapso tempore infinito fit $\cos s = -\frac{1}{n}$ et $\sin s = \frac{\sqrt{(nn-1)}}{n}$, eodemque casu evadit distantia $AB = \varrho$ infinita.

108. **Coroll. 2.** Elapso tempore infinito locus corporis in asymtotam incidet, et asymtotae utrinque ad axem hyperbolae inclinantur angulo, cujus cosinus est $= \frac{1}{n}$ et tangens $= \sqrt{(nn-1)}$. Est vero $\frac{f}{nn-1}$ semiaxis transversus hyperbolae et $\frac{f}{\sqrt{(nn-1)}}$ semiaxis conjugatus.

109. **Scholion.** Evolvimus ergo omnes species motuum, quibus duo corpora se mutuo attrahentia, siquidem fuerint sphaerica circumferri possunt; vidimus orbitam, quam alterum circa alterum describere spectatur, esse sectionem conicam. Huc quidem proxime accedunt orbitae, quas planetae primarii et cometae circa solem describere videntur, dum illi in ellipsibus circumferuntur, hi vero quasi in parabolis, etsi adhuc incertum est, utrum hyperbola penitus sit excludenda. Verumtamen planetas non exacte in orbitis ellipticis circa solem circumferri vel exinde patet, quod lineae absidum in coelo non quietae deprehenduntur. Duplex scilicet perturbatio eorum motum afficit, quarum altera a figura planetarum non sphaerica, altera ab attractione reliquorum corporum coelestium proficiscitur, quam investigationem deinceps sumus suscepturi. Ante autem juvabit hoc idem argumentum de motu duorum corporum sphaericorum per calculos variatos pertractasse. Cum enim totum negotium resolutione aequationum differentio-differentialium innitatur, plurimum intererit hujusmodi aequationes variis methodis tentari, quandoquidem hoc casu de successu certi sumus, quacunque methodo utamur, etiamsi forte, nisi solutio jam ante esset cognita, calculi evolutio nimis ardua videretur. His autem difficultatibus superatis, aditus ad sublimiores investigationes, quando plura duobus corpora proponuntur, facilior forsitan redderetur. In sequente ergo capite aliis quibusdam methodis determinationem motus duorum corporum sphaericorum aggrediamur.

Caput III.

Aliae investigationes motus duorum corporum sphaericorum.

110. **Problema.** (Fig. 179.) Dum corpora sphaerica A et B se mutuo attrahunt, hujus motum, qualis ex illo spectatur, referre ad planum quodcunque per corpus A ductum.

Solutio. Repraesentet tabula planum, ad quod motum corporis B referri oportet, quod jam tempore elapso $= t$ versetur in B, unde demisso ad planum propositum perpendiculo BY, et ex Y

ad rectam fixam AE normali YX, sint ternae coordinatae $AX = x$, $XY = y$, $YB = z$; ipsa distantia $AB = \rho = \sqrt{(xx + yy + zz)}$. Cum jam vis, qua B ad A urgetur, sit $= \frac{B(A+B)}{\rho\rho}$ (massas corporum per litteras A et B indicando), ea secundum directiones ternarum coordinatarum resoluta dabit vim in directione

$$AX = \frac{-B(A+B)x}{\rho^3},$$

$$XY = \frac{-B(A+B)y}{\rho^3},$$

$$YB = \frac{-B(A+B)z}{\rho^3},$$

unde sequentes aequationes elicimus sumendo elementum dt constans

$$ddx = \frac{-2g(A+B)x}{\rho^3}\cdot dt^2, \quad ddy = \frac{-2g(A+B)y}{\rho^3}\cdot dt^2, \quad ddz = \frac{-2g(A+B)z}{\rho^3}\cdot dt^2.$$

Hinc dt^2 eliminando colligimus

$$yddx - xddy = 0, \quad zddy - yddz = 0, \quad xddz - zddx = 0,$$

quarum quidem quaelibet in binis reliquis jam continetur, ita ut duas tractasse sufficiat. Inde ergo integrando obtinemus

$$ydx - xdy = Edt \quad \text{et} \quad zdy - ydz = Fdt, \quad \text{hincque}$$

$$F(ydx - xdy) + E(ydz - zdy) = 0,$$

quae per yy divisa et integrata dat

$$\frac{Fx}{y} + \frac{Ez}{y} + G = 0, \quad \text{seu} \quad Ez + Fx + Gy = 0,$$

ex qua liquet motum corporis B fieri in plano per punctum A transeunte. Cum igitur habeamus $Ez + Fx + Gy = 0$, ac praeterea has tres aequationes differentiales

$$Edt = ydx - xdy, \quad Fdt = zdy - ydz, \quad Gdt = xdz - zdx,$$

ob $xx + yy + zz = \rho\rho$ adipiscemur quadratis addendis

$$(EE + FF + GG)dt^2 = dx^2(\rho\rho - xx) + dy^2(\rho\rho - yy) + dz^2(\rho\rho - zz) - 2xydxdy - 2yzdydz - 2xzdxdz$$

et quia $xdx + ydy + zdz = \rho d\rho$, obtinebimus

$$(EE + FF + GG)\,dt^2 = \rho\rho\,(dx^2 + dy^2 + dz^2) - \rho\rho d\rho^2.$$

Verum si aequationum differentio-differentialium prima per $2dx$, secunda per $2dy$ et tertia per $2dz$ multiplicetur, summa erit

$$2dxddx + 2dyddy + 2dzddz = \frac{-4g(A+B)\rho d\rho}{\rho^3}\cdot dt^2,$$

cujus integrale, ob dt constans, est

$$dx^2 + dy^2 + dz^2 = Ddt^2 + \frac{4g(A+B)}{\rho}dt^2,$$

qui valor ob superiorem aequationem est

$$= dv^2 + \frac{(EE + FF + GG)\,dt^2}{vv},$$

ita ut per vv multiplicando habeamus

$$vv\,dv^2 = dt^2\,(Dvv + 4g\,(A + B)\,v - EE - FF - GG),$$

ideoque

$$dt = \frac{v\,dv}{\sqrt{(Dvv + 4g\,(A + B)\,v - EE - FF - GG)}}.$$

Superest ut reliquas variabiles x, y, z etiam per v determinemus. Cum igitur sit $z = \frac{-Fx - Gy}{E}$, habebimus

$$EEvv = EExx + EEyy + FFxx + 2FGxy + GGyy, \text{ hincque}$$

$$y = \frac{-FGx + E\sqrt{((EE + GG)\,vv - (EE + FF + GG)\,xx)}}{EE + GG}.$$

Statuamus brevitatis gratia

$$EE + FF + GG = HH, \quad \text{sitque} \quad \frac{Hx}{v\sqrt{(EE + GG)}} = \cos\omega, \quad \text{seu} \quad x = \frac{v\sqrt{(EE + GG)}}{H}\cos\omega,$$

$$\text{erit} \quad y = \frac{-FG\cos\omega + EH\sin\omega}{H\sqrt{(EE + GG)}}\cdot v \quad \text{et} \quad z = \frac{-EF\cos\omega - GH\sin\omega}{H\sqrt{(EE + GG)}}\cdot v.$$

Hinc erit $\frac{y}{x} = \frac{-FG\cos\omega + EH\sin\omega}{(EE + GG)\cos\omega}$ et $\frac{xdy - ydx}{xx} = \frac{EH\,d\omega}{(EE + GG)\cos^2\omega}$, ideoque

$$xdy - ydx = \frac{Evv\,d\omega}{H} = -E\,dt, \quad \text{ita ut sit} \quad d\omega = \frac{-H\,dt}{vv}.$$

Quaeratur ergo angulus ω, ut sit

$$\omega = -\int \frac{H\,dv}{v\sqrt{(Dvv + 4g\,(A + B)\,v - HH)}},$$

eritque $\sin\omega = \frac{HH - 2g\,(A + B)\,v}{v\sqrt{(DHH + 4gg\,(A + B)^2)}}$ et $\cos\omega = \frac{H\sqrt{(Dvv + 4g\,(A + B)\,v - HH)}}{v\sqrt{(DHH + 4gg\,(A + B)^2)}}$.

Invento hoc angulo ω, ad eum constantem angulum quemcunque adjicere licet, unde obtinebimus

$$\frac{x}{v} = \frac{(EE + GG)\cos(\omega + \delta)}{H\sqrt{(EE + GG)}},$$

$$\frac{y}{v} = \frac{-FG\cos(\omega + \delta) + EH\sin(\omega + \delta)}{H\sqrt{(EE + GG)}},$$

$$\frac{z}{v} = \frac{-EF\cos(\omega + \delta) - GH\sin(\omega + \delta)}{H\sqrt{(EE + GG)}},$$

ita ut jam omnes quantitates variabiles sint determinatae per eandem variabilem v.

111. Coroll. 1. Si ponatur $\frac{EH}{FG} = \operatorname{tang}\alpha$ et $\frac{GH}{EF} = \operatorname{tang}\gamma$, formulae posteriores transmutantur in has:

$$\frac{x}{v} = \frac{\sqrt{(EE + GG)}}{H}\cos(\omega + \delta), \quad \frac{y}{v} = \frac{-\sqrt{(EE + FF)}}{H}\cos(\omega + \delta + \alpha), \quad \frac{z}{v} = \frac{-\sqrt{(FF + GG)}}{H}\cos(\omega + \delta - \gamma),$$

quae formulae exprimunt cosinus angulorum, quibus recta AB ad ternas directiones principales inclinatur.

112. **Coroll. 2.** Si ducta AY, angulus XAY tanquam longitudo vocetur $= \varphi$, ob $\frac{y}{x} = \operatorname{tang} \varphi$, habebimus hanc longitudinis determinationem

$$\operatorname{tang} \varphi = \frac{-FG}{EE + GG} + \frac{EH \operatorname{tang}(\omega + \delta)}{EE + GG},$$

tum vero angulo YAB tanquam latitudine posito $= \psi$, erit

$$\sin \psi = \frac{-EF \cos(\omega + \delta) - GH \sin(\omega + \delta)}{H\sqrt{(EE + GG)}} = \frac{-\sqrt{(FF + GG)}}{H} \cos(\omega + \delta - \gamma).$$

113. **Coroll. 3.** Si recta $A☊$ fuerit intersectio plani, in quo corpus B movetur cum plano tabulae, motusque fiat in sensum EB, ita ut in ☊ sit nodus ascendens, erit pro longitudine hujus nodi $\operatorname{tang} EA☊ = \frac{-F}{G}$, et pro inclinatione planorum $\operatorname{tang} YNB = \frac{-\sqrt{(FF + GG)}}{E}$, seu $\cos YNB = \frac{-E}{H}$, ducta YN ad $A☊$ normali.

114. **Coroll. 4.** Si ponamus $\frac{F}{H} = \cos \alpha$, $\frac{G}{H} = \cos \beta$, $\frac{E}{H} = \cos \gamma$, ut sit

$$x \cos \alpha + y \cos \beta + z \cos \gamma = 0 \quad \text{et} \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1;$$

tum vero statuamus

$$\frac{x}{v} = \sin \alpha \cos(\omega + \zeta), \quad \frac{y}{v} = \sin \beta \cos(\omega + \eta), \quad \frac{z}{v} = \sin \gamma \cos(\omega + \vartheta),$$

anguli ζ, η, ϑ ita sunt comparati ut sit

$$\operatorname{tang}(\vartheta - \eta) = \frac{\cos \alpha}{\cos \beta \cos \gamma}, \quad \operatorname{tang}(\zeta - \vartheta) = \frac{\cos \beta}{\cos \alpha \cos \gamma}, \quad \operatorname{tang}(\eta - \zeta) = \frac{\cos \gamma}{\cos \alpha \cos \beta},$$

unde ordo in his formulis facilius perspicitur.

115. **Scholion.** Evidens est hanc solutionem ad superiorem perducere. Si enim angulus ☊AB ponatur $= \varphi$, qui est angulus, quem corpus B in sua orbita plana tempore t absolvit, erit $\frac{BN}{v} = \sin \varphi$; at est $\frac{z}{BN} = \sin YNB = \frac{\sqrt{(FF + GG)}}{H}$, ideoque

$$BN = \frac{Hz}{\sqrt{(FF + GG)}} \quad \text{et} \quad \sin \varphi = \frac{H}{\sqrt{(FF + GG)}} \cdot \frac{z}{v} = -\cos(\omega + \delta - \gamma) \quad \text{per § 111.}$$

Ergo $\sin \varphi = \sin(\omega + \delta - \gamma - 90^0)$, ita ut sit

$$\varphi = \omega + \text{Const. ac } d\varphi = d\omega, \quad \text{seu} \quad d\varphi = \frac{-H dv}{v\sqrt{(Dvv + 4G(A + B)v - HH)}},$$

quae aequatio cum supra inventa plane congruit. Signum enim $-$, quod supra erat $+$, ob signum radicalis ambiguum nihil turbat. Quin etiam poteramus ponere

$$\sin \varphi = + \cos(180^0 + \omega + \delta - \gamma) = \sin(-90^0 - \omega - \delta + \gamma),$$

unde deducitur $\varphi = \text{Const.} - \omega$ et $d\varphi = -d\omega$, eademque prorsus aequatio obtinetur. Ceterum cum hic motum corporis B etiam, qualis ex A spectatur, definiverimus, nunc in motus absolutos utriusque corporis inquiramus, quales scilicet ambo ex puncto quodam fixo visi apparerent; ac primo quidem ambos motus in eodem plano absolvi assumamus.

116. **Problema.** (Fig. 180.) Si duo corpora sphaerica se mutuo attrahentia A et B moveantur in eodem plano, definire eorum motum absolutum.

Solutio. Moveantur ambo corpora A et B, quorum massae iisdem litteris indicentur, in plano tabulae, in quo assumta recta fixa OV, in eaque puncto fixo O, ad quodvis tempus elapsum pro utroque corpore coordinatas orthogonales OX, XA et OP, PB assignari oportet. Ponamus ergo pro corpore A coordinatas $OX = X$, $XA = Y$, et ducta AL rectae fixae OV parallela, vocatisque distantia $AB = \rho$ et angulo $LAB = \varphi$, pro corpore B erunt coordinatae

$$OP = X + \rho \cos\varphi \quad \text{et} \quad PB = Y + \rho \sin\varphi.$$

Jam quia vis, qua corpora se mutuo attrahunt, est $= \frac{AB}{\rho\rho}$, corpus A sollicitabitur secundum directiones fixas

$$\text{sec. } OX \text{ vi} = \frac{AB}{\rho\rho}\cos\varphi; \qquad \text{sec. } XA \text{ vi} = \frac{AB}{\rho\rho}\sin\varphi;$$

corpus vero B sollicitabitur

$$\text{sec. } OP \text{ vi} = \frac{-AB}{\rho\rho}\cos\varphi; \qquad \text{sec. } PB \text{ vi} = \frac{-AB}{\rho\rho}\sin\varphi.$$

Sumto ergo elemento temporis dt constante, habebimus has quatuor aequationes:

$$\text{I. } ddX = \frac{2gBdt^2}{\rho\rho}\cos\varphi, \qquad \text{II. } ddY = \frac{2gBdt^2}{\rho\rho}\sin\varphi,$$

$$\text{III. } ddX + dd.\rho\cos\varphi = \frac{-2gAdt^2}{\rho\rho}\cos\varphi, \qquad \text{IV. } ddY + dd.\rho\sin\varphi = \frac{-2gAdt^2}{\rho\rho}\sin\varphi,$$

unde sublatis ddX et ddY supererunt hae duae aequationes

$$(1)\ dd.\rho\cos\varphi = \frac{-2g(A+B)\,dt^2}{\rho\rho}\cos\varphi, \qquad (2)\ dd.\rho\sin\varphi = \frac{-2g(A+B)\,dt^2}{\rho\rho}\sin\varphi,$$

quibus definitur motus respectivus corporis B, qualis spectatori in A posito esset appariturus, quippe qui motus per distantiam $AB = \rho$ et angulum $LAB = \varphi$ determinatur. Conveniuntque hae formulae perfecte cum iis, quas in superiori capite invenimus. Definito autem hoc motu respectivo, pro absoluto deinceps colligimus

$$(A+B)\,ddX + Bdd.\rho\cos\varphi = 0 \quad \text{et} \quad (A+B)\,ddY + Bdd.\rho\sin\varphi = 0,$$

ac proinde bis integrando

$$(A+B)\,X + B\rho\cos\varphi = Et + \mathfrak{E} \quad \text{et} \quad (A+B)\,Y + B\rho\sin\varphi = Ft + \mathfrak{F},$$

quibus motus uniformis communis centri inertiae corporum in directum declaratur. Ad motum ergo absolutum utriusque corporis cognoscendum primo motum respectivum investigari convenit, quod etsi jam in superiori capite est praestitum, solutionem tamen ex binis aequationibus hic expositis petamus. Ac primo quidem haec combinatio $(1).\rho\sin\varphi - (2).\rho\cos\varphi$ praebet

$$\rho\sin\varphi\, dd.\rho\cos\varphi - \rho\cos\varphi\, dd.\rho\sin\varphi = 0,$$

quae integrata dat $\quad v\sin\varphi\, d.v\cos\varphi - v\cos\varphi\, d.v\sin\varphi = -2Cdt,$

$$\text{seu} \quad vv\,d\varphi = Cdt.$$

Deinde ista combinatio $(1).d.v\cos\varphi + (2).d.v\sin\varphi$ praebet

$$d.v\cos\varphi.dd.v\cos\varphi + d.v\sin\varphi.dd.v\sin\varphi = \frac{-2g(A+B)\,dt^2}{vvv}(\cos\varphi\, d.v\cos\varphi + \sin\varphi\, d.v\sin\varphi)$$

$$= \frac{-2g(A+B)\,dt^2}{vv}.dv,$$

unde integrando impetramus

$$(d.v\cos\varphi)^2 + (d.v\sin\varphi)^2 = D + \frac{4g(A+B)\,dt^2}{v}, \quad \text{sive} \quad dv^2 + vv\,d\varphi^2 = D\,dt^2 + \frac{4g(A+B)\,dt^2}{v}.$$

Quare cum ex illa sit $dt = \frac{vv\,d\varphi}{C}$, fiet

$$CC\,dv^2 + CCvv\,d\varphi^2 = Dv^4\,d\varphi^2 + 4g(A+B)v^3\,d\varphi^2,$$

$$\text{hincque} \qquad d\varphi = \frac{C\,dv}{v\sqrt{(Dv^2 + 4g(A+B)v - CC)}}$$

$$\text{atque} \qquad dt = \frac{v\,dv}{\sqrt{(Dv^2 + 4g(A+B)v - CC)}}.$$

Definitis autem ad tempus t quantitatibus v et φ, ex superioribus formulis colligentur coordinatae X et Y pro corpore A, ex quibus hujus corporis motus absolutus innotescit, indeque etiam corporis alterius B.

117. **Coroll. 1.** Si J sit commune centrum inertiae amborum corporum, erit

$$(A+B)\,OK = A.OX + B.OP = (A+B)\,X + Bv\cos\varphi$$

$$\text{et} \quad (A+B)\,KJ = A.XA + B.PB = (A+B)\,Y + Bv\sin\varphi,$$

unde in superioribus formulis E est celeritas ejus in directione OV, et F in directione KJ.

118. **Coroll. 2.** Positis ergo $E = 0$ et $F = 0$, commune centrum inertiae J quiescet. Ac punctum O in eo ipso accipiamus, insuper constantes $\mathfrak{E}$ et $\mathfrak{F}$ evanescunt, eritque tum

$$OX = X = \frac{-Bv\cos\varphi}{A+B} \quad \text{et} \quad XA = Y = \frac{-Bv\sin\varphi}{A+B}.$$

119. **Coroll. 3.** Cum in superiori capite per anomaliam veram s has determinationes invenerimus:

$$v = \frac{f}{1+n\cos s}, \quad \varphi = s + \alpha \quad \text{et} \quad t = \frac{f\sqrt{f}}{\sqrt{2g(A+B)}}\int\frac{ds}{(1+n\cos s)^2},$$

habebimus in genere pro curva, quam corpus A motu absoluto describit, constantibus parum immutatis,

$$OX = X = \mathfrak{E} + E\int\frac{ds}{(1+n\cos s)^2} - \frac{Bf\cos(s+\alpha)}{(A+B)(1+n\cos s)},$$

$$XA = Y = \mathfrak{F} + F\int\frac{ds}{(1+n\cos s)^2} - \frac{Bf\sin(s+\alpha)}{(A+B)(1+n\cos s)}.$$

120. **Coroll. 4.** Pro curva vero, quam alterum corpus B motu absoluto describit, erunt coordinatae

$$OP = X + v\cos\varphi = \mathfrak{E} + E\int\frac{ds}{(1+n\cos s)^2} + \frac{Af\cos(s+a)}{(A+B)(1+n\cos s)},$$

$$PB = Y + v\sin\varphi = \mathfrak{F} + F\int\frac{ds}{(1+n\cos s)^2} + \frac{Af\sin(s+a)}{(A+B)(1+n\cos s)}.$$

Unde patet casu, quo $E = 0$ et $F = 0$, utramque curvam fore sectionem conicam.

121. **Scholion.** Hinc perspicitur egregius consensus inter ambas methodos, quibus sum usus ad motus binorum corporum determinandos, ac simul patet, utramque methodum ita inter se cohaerere, ut determinatio motus respectivi praecipuam partem in motus absoluti investigatione constituat. Haec methodus scilicet latius patet, quam illa, cum non solum motum respectivum perinde ac illa patefaciat, sed etiam motum absolutum utriusque corporis declaret, atque hoc quidem ita, ut ratio motuum absolutorum facillime e calculo eliminetur, totumque negotium ad motus respectivi determinationem perducatur. Hoc enim cognito nihil aliud superest, nisi ut motus communis centri inertiae, qui semper est uniformis secundum lineam rectam, in computum introducatur. Quare etiam in investigatione motus plurium corporum se mutuo attrahentium semper sufficit motus respectivos, qui spectatori in uno eorum collocato sint apparituri, determinasse. Etsi enim hic unum corpus tanquam quiescens consideratur, tamen facile est deinceps toti systemati ejusmodi motum mente saltem inducere, quo commune centrum inertiae vel ad quietem vel motum uniformem rectilineum redigatur, hocque modo ad motuum absolutorum cognitionem pervenietur. Istud etiam eo clarius patebit ex sequente problemate, ubi motus binorum corporum, quando non in eodem plano absolvuntur, sum evoluturus.

122. **Problema.** Si duo corpora sphaerica se mutuo attrahentia ita moveantur, ut motus eorum non in eodem plano absolvatur, definire utriusque corporis motum absolutum.

Solutio. (Fig. 181) Sint jam elapso tempore $= t$ ambo corpora in A et B, quorum massae iisdem litteris A et B indicentur. Referantur eorum loca ad ternas directiones fixas inter se normales OE, OF, OG, quibus constituantur pro utroque parallelae coordinatae, quas pro A vocemus $OX = X$, $XY = Y$ et $YA = Z$. Pro corpore autem B statuamus primo distantiam $AB = v$, tum vero ejus inclinatio ad ternas illas directiones fixas OE, OF, OG indicetur angulis ζ, η, ϑ, ut sit $\cos^2\zeta + \cos^2\eta + \cos^2\vartheta = 1$, hincque coordinatae pro B erunt $OP = X + v\cos\zeta$, $PQ = Y + v\cos\eta$, $QB = Z + v\cos\vartheta$, seu posito $v\cos\zeta = x$, $v\cos\eta = y$, $v\cos\vartheta = z$, ut sit $vv = xx + yy + zz$, habebimus $OP = X + x$, $PQ = Y + y$, $QB = Z + z$. Cum jam vis attractrix secundum AB sit $= \frac{AB}{vv}$, corpus A ab ea sollicitatur

$$\text{secundum } OE \text{ vi} = \frac{AB}{vv}\cos\zeta,\quad \text{sec. } OF \text{ vi} = \frac{AB}{vv}\cos\eta,\quad \text{sec. } OG \text{ vi} = \frac{AB}{vv}\cos\vartheta,$$

corpus vero B his viribus

$$\text{sec. } OE \text{ vi} = \frac{-AB}{vv}\cos\zeta,\quad \text{sec. } OF \text{ vi} = \frac{-AB}{vv}\cos\eta,\quad \text{sec. } OG \text{ vi} = \frac{-AB}{vv}\cos\vartheta,$$

unde principia accelerationis suppeditabunt has aequationes

$$ddX = \frac{2gB}{vv}dt^2\cos\zeta, \qquad ddY = \frac{2gB}{vv}dt^2\cos\eta, \qquad ddZ = \frac{2gB}{vv}dt^2 \ldots$$

$$ddX + ddx = \frac{-2gA}{vv}dt^2\cos\zeta, \quad ddY + ddy = \frac{-2gA}{vv}dt^2\cos\eta, \quad ddZ + ddz = \frac{-2gA}{vv}dt^2 \ldots$$

Hic quantitates X, Y, Z referuntur ad motum absolutum corporis A, at x, y, z ad motum ... spectivum, quo corpus B ex A spectatum moveri cernitur. Pro hoc ergo colligimus:

$$\text{I.}\quad ddx = \frac{-2g(A+B)dt^2\cos\xi}{vv} = \frac{-2g(A+B)x\,dt^2}{v^3},$$

$$\text{II.}\quad ddy = \frac{-2g(A+B)dt^2\cos\eta}{vv} = \frac{-2g(A+B)y\,dt^2}{v^3},$$

$$\text{III.}\quad ddz = \frac{-2g(A+B)dt^2\cos\vartheta}{vv} = \frac{-2g(A+B)z\,dt^2}{v^3},$$

ex quibus cum aequatione $xx + yy + zz = vv$ omnes quantitates x, y, z et v ad tempus ... minari oportet. Inde autem primo has aequationes integrabiles deducimus

$$yddx - xddy = 0, \quad zddy - yddz = 0, \quad xddz - zddx = 0,$$

quae integratae dant

$$ydx - xdy = Edt, \quad zdy - ydz = Fdt, \quad xdz - zdx = Gdt.$$

Quare cum sit $F(ydx - xdy) = E(zdy - ydz)$, per yy dividendo nanciscemur $\frac{Fx}{y} = \frac{-Ez}{y} \ldots$ Similique modo ob $G(zdy - ydz) = F(xdz - zdx)$, per zz dividendo adipiscimur $\frac{Gy}{z} = \frac{-Fx}{z} \ldots$ ex quibus conjunctim deducimus $Fx + Gy + Ez = 0$, qua aequatione motus corporis B ex ... tatus in eodem plano fieri indicatur.

Porro si primam per $2dx$, secundam per $2dy$ et tertiam per $2dz$ multiplicemus, ob $xdx + ydy + zdz = vdv$, summa erit

$$2dxddx + 2dyddy + 2dzddz = \frac{-4g(A+B)dv}{vv}dt^2,$$

cujus integrale ob dt constans dat

$$dx^2 + dy^2 + dz^2 = Ddt^2 + \frac{4g(A+B)dt^2}{v},$$

ex qua ope aequationum

$$Fx + Gy + Ez = 0, \quad xx + yy + zz = vv, \quad ydx - xdy = Edt, \quad zdy - ydz = Fdt, \quad xdz - zdx = Gdt$$

eadem solutio deducitur, quam jam supra dedimus. Denique inventis variabilibus x, y, z ... respectivum spectantibus, ex iis pro motu absoluto corporis A colliguntur coordinatae X, Y, Z ... has aequationes

$$X = \frac{-Bx + \mathfrak{E}t + \mathfrak{e}}{A+B}, \quad Y = \frac{-By + \mathfrak{F}t + \mathfrak{f}}{A+B}, \quad Z = \frac{-Bz + \mathfrak{G}t + \mathfrak{g}}{A+B}.$$

123. **Coroll. 1.** Ex aequationibus differentialibus

$$ydx - xdy = Edt, \quad zdy - ydz = Fdt, \quad xdz - zdx = Gdt,$$

... immediate colligitur, multiplicando primam per z, secundam per x, et tertiam per z, ... sumendo:

$$0 = Ezdt + Fxdt + Gydt, \quad \text{hincque} \quad Ez + Fx + Gy = 0.$$

124. **Coroll. 2.** Eaedem aequationes differentiales quadratae et additae, posito

$$EE + FF + GG = HH, \quad \text{praebent}$$

$$HHdt^2 = vv(dx^2 + dy^2 + dz^2) - vvdv^2, \quad \text{ideoque} \quad dx^2 + dy^2 + dz^2 = dv^2 + \frac{HHdt^2}{vv}.$$

125. **Coroll. 3.** Si illae aequationes differentiales combinentur cum hac

$$xdx + ydy + zdz = vdv,$$

... dx, dy et dz inde ita definiuntur, ut sit

$$dx = \frac{xdv}{v} + \frac{(Ey - Gz)\,dt}{vv}, \quad dy = \frac{ydv}{v} + \frac{(Fz - Ex)\,dt}{vv}, \quad dz = \frac{zdv}{v} + \frac{(Gx - Fy)\,dt}{vv}.$$

126. **Coroll. 4.** Cum autem sit $Ez + Gy = -Fx$, si ponamus $Ey - Gz = p$, erit quadratis ...

$$(EE + GG)(yy + zz) = (EE + GG)(vv - xx) = FFxx + pp,$$

... ob $FF + GG + EE = HH$, erit $p = Ey - Gz = \sqrt{((EE + GG)vv - HHxx)}$, hincque

$$\frac{vdx - xdv}{vv} = d \cdot \frac{x}{v} = \frac{dt}{vv}\sqrt{\left(EE + GG - \frac{HHxx}{vv}\right)}.$$

127. **Scholion.** Hae investigationes non solum inserviunt motui absoluto definiendo, etsi in ... parum interest eum nosse, sed imprimis eas ideo hic attulimus, ut intelligatur, quo... hujusmodi solutiones, ubi plures aequationes differentio-differentiales occurrunt, tractari con... Cum enim in sequentibus omnia a resolutione talium aequationum pendeant, in hujusmodi ... maxime juvabit vires analyseos exercuisse, unde haec tractatio utilitate non caritura videtur. ... ergo motu duorum corporum sphaericorum, quod quidem argumentum jam passim satis ... est pertractatum, antequam casum trium corporum aggrediamur, in motum duorum cor... non sphaericorum inquiramus, ut pateat, quantum discrimen a defectu sphaericitatis proficis... Cum enim tam solis quam planetarum corpora a figura sphaerica recedant, jam ob hanc causam irregularitates quaedam se motui, qui per regulas consuetas in hypothesi corporum ...corum determinatur, admiscebunt, quarum cognitio eo magis est necessaria, ne phaenomena ...da actioni aliorum corporum tribuantur. Hic vero alteri tantum corpori figuram a sphae... diversam assignabimus, alterum perfecte sphaericum relinquentes; si enim ambo non fuerint ... primo alterum tanquam sphaericum spectetur, tum vero alterum, quo facto ex combina... phaenomenorum solutio haud difficulter colligetur, praecipue cum viderimus a defectu figurae ... motus parum perturbari.

Caput IV.

De motu duorum corporum, quorum alterum tantum est sphaericum.

128. **Problema.** Si corpus sphaericum moveatur circa corpus figura quacunque praedi- quod omni motu rotatorio careat, invenire aequationes, quibus ejus motus determi-

Solutio. Cum quaestio sit de motu respectivo corporis sphaerici, alterum corpus non sphae- ricum in quiete considerabimus, quoniam ipsi etiam omnem motum rotatorium adimimus. Sit hujus corporis centrum inertiae in J (fig. 172), ejusque axes principales JA, JB, JC, quorum re- spectu sint momenta inertiae Maa, Mbb, Mcc, denotante M massam hujus corporis, quod tanquam quiescens spectamus. Nunc autem elapso tempore $=t$, alterius corporis sphaerici centrum versetur in H, ejusque massa vocetur $=N$, unde ad planum binis axibus principalibus prioris corporis JA, JB contentum demittatur perpendiculum HG, et ex G ad axem JA ducatur normalis GF, ut habea- tur pro ejus loco ternae coordinatae $JF=x$, $FG=y$ et $GH=z$; distantia autem ipsa JH vocetur $=v$. Quodsi jam huc denominationes § 38 accommodemus, erit $h=v$, $\cos\alpha=\frac{x}{v}$, $\cos\beta=\frac{y}{v}$, $\cos\gamma=\frac{z}{v}$, unde corpus N in H sequentibus tribus viribus secundum directiones $H\alpha$, $H\beta$, $H\gamma$ axi- bus principalibus parallelas sollicitatur

$$\text{secundum } H\alpha=\frac{MNx}{v^3}\left(1+\frac{3aa}{2vv}\left(3-\frac{5xx}{vv}\right)+\frac{3bb}{2vv}\left(1-\frac{5yy}{vv}\right)+\frac{3cc}{2vv}\left(1-\frac{5zz}{vv}\right)\right),$$

$$H\beta=\frac{MNy}{v^3}\left(1+\frac{3bb}{2vv}\left(3-\frac{5yy}{vv}\right)+\frac{3cc}{2vv}\left(1-\frac{5zz}{vv}\right)+\frac{3aa}{2vv}\left(1-\frac{5xx}{vv}\right)\right),$$

$$H\gamma=\frac{MNz}{v^3}\left(1+\frac{3cc}{2vv}\left(3-\frac{5zz}{vv}\right)+\frac{3aa}{2vv}\left(1-\frac{5xx}{vv}\right)+\frac{3bb}{2vv}\left(1-\frac{5yy}{vv}\right)\right).$$

A paribus autem viribus, sed contrario modo applicatis corpus M ad corpus in H sollicitatur, quae cum denuo, ob motum respectivum, contrario modo ad corpus in H sint transferendae et in ratione massarum M ad N mutandae, motus respectivus corporis in H sequentibus tribus aequationibus dif- ferentio-differentialibus exprimetur, sumto elemento temporis dt constante:

$$ddx=\frac{-2g(M+N)x\,dt^2}{v^3}\left(1+\frac{3(3aa+bb+cc)}{2vv}-\frac{15(aaxx+bbyy+cczz)}{2v^4}\right),$$

$$ddy=\frac{-2g(M+N)y\,dt^2}{v^3}\left(1+\frac{3(aa+3bb+cc)}{2vv}-\frac{15(aaxx+bbyy+cczz)}{2v^4}\right),$$

$$ddz=\frac{-2g(M+N)z\,dt^2}{v^3}\left(1+\frac{3(aa+bb+3cc)}{2vv}-\frac{15(aaxx+bbyy+cczz)}{2v^4}\right),$$

ubi notandum esse $vv=xx+yy+zz$. Hinc autem primo colligimus

$$\frac{ddx}{x}-\frac{ddy}{y}=\frac{-2g(M+N)\,dt^2}{v^3}\cdot\frac{3(aa-bb)}{vv},$$

$$\frac{ddy}{y}-\frac{ddz}{z}=\frac{-2g(M+N)\,dt^2}{v^3}\cdot\frac{3(bb-cc)}{vv},$$

$$\frac{ddz}{z}-\frac{ddx}{x}=\frac{-2g(M+N)\,dt^2}{v^3}\cdot\frac{3(cc-aa)}{vv},$$

Hinc porro

$$\frac{yddx - xddy}{(aa-bb)xy} = \frac{zddy - yddz}{(bb-cc)yz} = \frac{xddz - zddx}{(cc-aa)xz},$$

seu $\quad (aa-bb)\,xyddz + (bb-cc)\,yzddx + (cc-aa)\,xzddy = 0,$

quam autem immediate ulterius reducere non licet.

Verum ex tribus illis primis formulis, si brevitatis gratia ponamus $a^2x^2 + b^2y^2 + c^2z^2 = f^2u^2$, obtinebimus hanc aequationem integrabilem

$$dxddx + dyddy + dzddz = \frac{-2g\,(M+N)\,dt^2}{v^3}\left(v dv + \frac{3\,(aa+bb+cc)\,dv}{2v} + \frac{3\,ffudu}{vv} - \frac{15\,ffuudv}{2v^3}\right)$$

$$= -2g(M+N)\,dt^2\left(\frac{dv}{vv} + \frac{3\,(aa+bb+cc)\,dv}{2v^4} + \frac{3\,ffudu}{v^5} - \frac{15\,ffuudv}{2v^6}\right).$$

Hujus enim integrale est

$$dx^2 + dy^2 + dz^2 = Ddt^2 + 4g\,(M+N)\,dt^2\left(\frac{1}{v} + \frac{aa+bb+cc}{2v^3} - \frac{3\,ffuu}{2v^5}\right).$$

Cum autem praeterea aliae integrationes non adsint, hinc solutionem in quantitatibus finitis expressam deducere non licet.

129. **Coroll. 1.** Ob tres variabiles x, y, z per tempus t determinandas requiruntur tres aequationes, unde cum integrali postremo loco inventa adhuc duas conjungi oportet, ac perinde est quaenam ad hunc finem eligantur.

130. **Coroll. 2.** Loco alterius harum commodissime accipi videtur haec, in quam ternae variabiles x, y et z aequaliter ingrediuntur

$$(aa-bb)\,xyddz + (bb-cc)\,yzddx + (cc-aa)\,xzddy = 0,$$

quae etiam ad hanc formam reducitur

$$aax\,(yddz - zddy) + bby\,(zddx - xddz) + ccz\,(xddy - yddx) = 0.$$

131. **Coroll. 3.** Loco tertiae vero aequationis pro lubitu una ex his tribus accipietur

$$yddx - xddy = \frac{-6g\,(aa-bb)\,(M+N)\,xydt^2}{v^5},$$

$$zddy - yddz = \frac{-6g\,(bb-cc)\,(M+N)\,yzdt^2}{v^5},$$

$$xddz - zddx = \frac{-6g\,(cc-aa)\,(M+N)\,xzdt^2}{v^5}.$$

132. **Scholion 1.** Quomodocunque autem hae aequationes cum ista

$$dx^2 + dy^2 + dz^2 = Ddt^2 + 4g\,(M+N)\,dt^2\left(\frac{1}{v} + \frac{aa+bb+cc}{2v^3} - \frac{3\,ffuu}{2v^5}\right)$$

posito brevitatis gratia $aaxx + bbyy + cczz = ffuu$ combinentur, solutio problematis maximis difficultatibus involvitur. Quare cum problema latissime pateat ob figuram quamcunque, quam corpori quiescenti tribuimus, casus magis particulares contemplemur; ac primo quidem statim patet, si duo corporis M momenta principalia fuerint inter se aequalia, difficultates illas maximam partem ... cere. Statuamus ergo momenta inertiae respectu axium JA et JB inter se aequalia, seu bb ... atque evidens est in hac hypothesi perinde esse, sive corpus M quiescat, sive ei motus gyr... quicunque circa axem tertium JC tribuatur, quoniam omnia momenta respectu axium in plano quod tanquam quiescens spectamus, sumtorum, sunt inter se aequalia. Pro hoc ergo casu mo... respectivum alterius corporis sphaerici N investigemus.

133. **Scholion 2.** Interim tamen conatus exposuisse juvabit, qui forte aliquando ad solu... nem producere valeant. Faciamus statim has substitutiones

$$x = pv, \quad ydx - xdy = ldt, \quad 2g(aa - bb)(M + N) = A,$$
$$y = qv, \quad zdy - ydz = mdt, \quad 2g(bb - cc)(M + N) = B,$$
$$z = rv, \quad xdz - zdx = ndt, \quad 2g(cc - aa)(M + N) = C,$$

ex quibus pro novis litteris concludimus has relationes

$$pp + qq + rr = 1, \quad lz + mx + ny = 0, \quad \text{seu} \quad lr + mp + nq = 0,$$

tum $A + B + C = 0$, item $Acc + Baa + Cbb = 0$. Porro ob $yddx - xddy = dldt$, habebimus

$$dl = \frac{-3Apqdt}{v^3}, \quad dm = \frac{-3Bqrdt}{v^3}, \quad dn = \frac{-3Cprdt}{v^3}.$$

Deinde cum sit $ydx - xdy = vv(qdp - pdq)$ nanciscimur

$$qdp - pdq = \frac{ldt}{vv}, \quad rdq - qdr = \frac{mdt}{vv}, \quad pdr - rdp = \frac{ndt}{vv},$$

unde fit $\frac{(lq - nr)dt}{vv} = qqdp - pqdq - prdr + rrdp = dp$, quia est $-qdq - rdr = pd$... $pp + qq + rr = 1$. Sicque erit

$$dp = \frac{(lq - nr)dt}{vv}, \quad dq = \frac{(mr - lp)dt}{vv}, \quad dr = \frac{(np - mq)dt}{vv};$$

atque hinc porro colligitur $rdl + pdm + qdn = 0$, $mdp + ndq + ldr = 0$; tum vero etiam

$$ccrdl + aapdm + bbqdn = 0, \quad \text{ideoque} \quad Cpdm = Bqdn, \quad Crdl = Aqdn, \quad Brdl = Apdm$$

$$\text{sive} \quad \frac{rdl}{A} = \frac{pdm}{B} = \frac{qdn}{C} = \frac{-3pqrdt}{v^3}.$$

Ex assumtis autem aequationibus obtinemus

$$dt^2(ll + mm + nn) = vv(dx^2 + dy^2 + dz^2) - vvdv^2,$$

ita ut nostra aequatio integralis futura sit

$$dv^2 + \frac{(ll + mm + nn)dt^2}{vv} = Ddt^2 + 2g(M + N)dt^2\left(\frac{2}{v} + \frac{aa + bb + cc}{v^3} - \frac{3(aapp + bbqq + ccrr)}{v^3}\right)$$

in qua, quia quantitates $ll + mm + nn$ et $aapp + bbqq + ccrr \ldots\ldots$, investigemus per formulas superiores earum differentialia. Reperiemus ergo

$$ldl + mdm + ndn = \frac{-3dt}{v^3}(Alpq + Bmqr + Cnpr) \quad \text{et}$$

$$aapdp + bbqdq + ccrdr = \frac{dt}{vv}((aa - bb)\,lpq + (bb - cc)\,mqr + (cc - aa)\,npr),$$

Jam ob $aa - bb = \frac{A}{2g(M+N)}$ erit

$$aapdp + bbqdq + ccrdr = \frac{dt}{2g(M+N)v^2}(Alpq + Bmqr + Cnpr),$$

ita ut sit

$$aapdp + bbqdq + ccrdr = \frac{-v(ldl + mdm + ndn)}{6g(M+N)}.$$

Hinc etiam differentio-differentialia primitiva definire possumus, cum enim sit

$$dx = pdv + vdp = pdv + \frac{(lq - nr)\,dt}{v},$$

erit

$$ddx = pddv + \frac{(lq - nr)\,dt\,dv}{vv} - \frac{(lq - nr)\,dt\,dv}{vv} + \frac{dt}{v}\,d.(lq - nr).$$

Est vero

$$d(lq - nr) = \frac{-3Apqqdt + 3Cprrdt}{v^3} + \frac{dt}{vv}(lmr - llp - nnp + mnq);$$

quae ob $lr + nq = -mp$ abit in

$$d(lq - nr) = \frac{-3pdt}{v^3}(Aqq - Crr) - \frac{pdt}{vv}(ll + mm + nn),$$

ita ut sit

$$ddx = pddv - \frac{pdt^2}{v^3}(ll + mm + nn) - \frac{3pdt^2}{v^4}(Aqq - Crr),$$

quae expressio aequalis est isti

$$\frac{-2g(M+N)\,pdt^2}{v^2}\left(1 + \frac{3(3aa + bb + cc)}{2vv} - \frac{15(aapp + bbqq + ccrr)}{2vv}\right).$$

Cum jam sit

$$Aqq - Crr = 2g(M+N)(aaqq - bbqq - ccrr + aarr) = 2g(M+N)(aa - aapp - bbqq - ccrr),$$

erit

$$ddv = \frac{dt^2(ll + mm + nn)}{v^3} - \frac{2g(M+N)\,dt^2}{v^2}\left(1 + \frac{3(aa + bb + cc) - 9(aapp + bbqq + ccrr)}{2vv}\right),$$

quae, aequationem integralem per dv multiplicando, facile reducitur.

En ergo octo variabiles t, v, l, m, n, p, q, r, quas determinari oportet ope harum aequationum:

1. $pp + qq + rr = 1$, 2. $lr + mp + nq = 0$

3. $dp = \frac{(lq - nr)\,dt}{vv}$, 6. $dl = \frac{-6g(aa - bb)(M+N)\,pq\,dt}{v^3}$

4. $dq = \frac{(mr - lp)\,dt}{vv}$, 7. $dm = \frac{-6g(bb - cc)(M+N)\,qr\,dt}{v^3}$

5. $dr = \frac{(np - mq)\,dt}{vv}$, 8. $dn = \frac{-6g(cc - aa)(M+N)\,pr\,dt}{v^3}$

9. $rdl + pdm + qdn = 0$, 10. $ldr + mdp + ndq = 0$

11. $ccrdl + aapdm + bbqdn = 0$,

cum quibus aequationem vel differentio-differentialem $dd\varrho$, vel integralem inde natam combin[are] oportet.

134. **Problema.** (Fig. 172). Si corpus M, quod ut quiescens spectatur, habeat momen[ta] inertiae principalia respectu axium JA et JB aequalia, idque sive quiescat, sive circ[a] axem tertium JC gyretur, definire ejus respectu motum alterius corporis sphaerici N.

Solutio. Retentis omnibus denominationibus, quas in problemate praecedente constituimus, [p]o[sito] $bb = aa$, aequatio nostra integralis erit

$$dx^2 + dy^2 + dz^2 = Ddt^2 + 4g(M+N)\,dt^2\left(\frac{1}{\varrho} + \frac{2aa+cc}{2\varrho^3} - \frac{3aa(xx+yy) - 3cczz}{2\varrho^5}\right).$$

Praeterea vero habebimus has duas

$$yddx - xddy = 0 \quad \text{et} \quad xddz - zddx = \frac{-6g(cc-aa)(M+N)\,xz\,dt^2}{\varrho^5}$$

existente $xx + yy + zz = \varrho\varrho$, quarum illa integrata praebet $ydx - xdy = Edt$. Statuamus praeter[ea] $zdy - ydz = qdt$ et $xdz - zdx = rdt$, eritque

$$dq = \frac{6g(cc-aa)(M+N)\,yz\,dt}{\varrho^5} \quad \text{et} \quad dr = \frac{-6g(cc-aa)(M+N)\,xz\,dt}{\varrho^5},$$

ita ut sit $xdq + ydr = 0$. Tum vero ex illis tribus formulis colligimus $Ez + qx + ry = 0$, atque insuper $(EE + qq + rr)\,dt^2 = \varrho\varrho(dx^2 + dy^2 + dz^2) - \varrho\varrho\,d\varrho^2$, hincque

$$dx^2 + dy^2 + dz^2 = d\varrho^2 + \frac{(EE+qq+rr)\,dt^2}{\varrho\varrho}.$$

Ex aequationibus $Ez + qx + ry = 0$ et $xx + yy + zz = \varrho\varrho$ concludimus

$$x = \frac{-Eqz + r\sqrt{((qq+rr)\varrho\varrho - (EE+qq+rr)zz)}}{qq+rr}, \quad y = \frac{-Erz - q\sqrt{((qq+rr)\varrho\varrho - (EE+qq+rr)zz)}}{qq+rr}.$$

Deinde vero habemus $dz = \frac{zd\varrho}{\varrho} - \frac{(qy - rx)dt}{\varrho\varrho}$, ideoque

$$dz = \frac{zd\varrho}{\varrho} + \frac{dt}{\varrho\varrho}\sqrt{((qq+rr)\varrho\varrho - (EE+qq+rr)zz)} \quad \text{et} \quad d\cdot\frac{z}{\varrho} = \frac{dt}{\varrho\varrho}\sqrt{\left(qq + rr - \frac{(EE+qq+rr)zz}{\varrho\varrho}\right)}$$

Aequatio autem prima hinc reducitur ad hanc formam

$$d\varrho^2 = Ddt^2 - \frac{(EE+qq+rr)\,dt^2}{\varrho\varrho} + 4g(M+N)\,dt^2\left(\frac{1}{\varrho} + \frac{(cc-aa)(\varrho\varrho - 3zz)}{2\varrho^5}\right).$$

Ponamus $z = u\varrho$, $q = s\cos\omega$ et $r = s\sin\omega$, eritque

$$d\varrho^2 = Ddt^2 - \frac{(EE+ss)\,dt^2}{\varrho\varrho} + 4g(M+N)\,dt^2\left(\frac{1}{\varrho} + \frac{(cc-aa)(1-3uu)}{2\varrho^3}\right)$$

atque $du = \frac{dt}{\varrho\varrho}\sqrt{(ss - (EE+ss)uu)}$; tum vero

$$x = \frac{-E\cos\omega + \sin\omega\sqrt{(ss-(EE+ss)uu)}}{s}\cdot\varrho, \quad y = \frac{-E\sin\ \ -\cos\omega\sqrt{(ss-(EE+ss)uu)}}{s}\cdot\varrho,$$

unde differentialia dq et dr supra definita dant

$$ds = -6g\,(cc - aa)\,(M + N)\cdot\frac{udt}{sv^3}\sqrt{(ss - (EE + ss)\,uu)},$$

$$sd\omega = +6g\,(cc - aa)\,(M + N)\cdot\frac{Eudt}{sv^3}.$$

Eliminato ergo dt primo pro determinatione harum trium quantitatum v, u, s hae duae aequationes obtinentur:

$$vv\,(ss - (EE + ss)\,uu) = du^2\Big(Dv^4 - (EE + ss)\,vv + 4g\,(M + N)\,v\,\big(v^2 + \tfrac{1}{2}(cc - aa)\,(1 - 3uu)\big)\Big)$$

$$\text{et}\quad sds = -6g\,(cc - aa)\,(M + N)\,\frac{udu}{v},$$

quae si resolvi possent, ex iis deinceps angulus ω et tempus t facile determinaretur. Sed vereor ne omnis labor hic nequicquam consumatur.

135. **Scholion.** Neque ergo hunc casum, etiamsi in suo genere facilis videatur, calculus expedire sinit. Verum si ponamus corpus B in ipso plano AJB, in quod axes principales aequalia momenta inertiae habentes incidunt, moveri, calculi difficultates superare licet, qui casus propterea meretur, ut omni cura evolvatur. Cum autem corpus M ita comparatum accipiatur, ut bina momenta inertiae, quae axibus JA et JB respondent, sint inter se aequalia, ei quasi unicus axis JC relinquitur, quoniam omnes axes in plano AJB assumti pari proprietate sunt praediti, sectionem per hoc planum factam tanquam aequatorem corporis spectare poterimus, praecipue cum corpori motum rotatorium quemcunque circa axem JC tribuere liceat. Quomodocunque scilicet corpus M circa axem JC gyretur, si alterum corpus B in ipso ejus aequatoris plano AJB moveatur, motum ejus calculo definire poterimus, id quod in sequente problemate praestabimus.

136. **Problema.** (Fig. 182.) Si corpus M, momenta inertiae respectu axium JA et JB aequalia habens, utcunque gyretur circa axem JC, alterumque corpus N in ipso illius plano aequatoris AJB moveatur, hujus motum respectivum definire.

Solutio. Cum sit $bb = aa$, omnia momenta inertiae ad axes in plano aequatoris sumtos relata sunt $= Maa$, momentum inertiae autem respectu axis $JC = Mcc$, circa quem corpus gyratur. Deinde ob applicatam $z = 0$, si corpus N in plano aequatoris tempore $= t$ confecerit arcum AN, ponamusque coordinatas $JX = x$, $XN = y$ et distantiam $JN = v$, ut sit $xx + yy = vv$, motus quaesitus his duabus aequationibus continetur

$$dx^2 + dy^2 = Ddt^2 + 4g\,(M + N)\,dt^2\left(\frac{1}{v} + \frac{2aa + cc}{2v^3} - \frac{3aa}{2v^3}\right)\quad \text{et}\quad ydx - xdy = Edt.$$

Cum ergo sit $ydy + xdx = vdv$, erit $EEdt^2 + vvdv^2 = (yy + xx)\,(dx^2 + dy^2) = vv\,(dx^2 + dy^2)$, ideoque

$$dv^2 + \frac{EEdt^2}{vv} = Ddt^2 + 4g\,(M + N)\,dt^2\left(\frac{1}{v} + \frac{cc - aa}{2v^3}\right)\quad \text{et}$$

$$dt = \frac{vdv}{\sqrt{\left(Dvv + 4g\,(M + N)\,v - EE + \dfrac{2g\,(cc - aa)\,(M + N)}{v}\right)}}.$$

Praeterea vero posito angulo $AJN=\varphi$, ut sit $x=v\cos\varphi$ et $y=v\sin\varphi$, erit $ydx-xdy=-vvd\varphi$, hincque sumto E negativo, $d\varphi=\frac{Edt}{vv}$, ac propterea

$$d\varphi=\frac{Edv}{v\sqrt{\left(Dvv+4g(M+N)v-EE+\frac{2g(cc-aa)}{v}(M+N)\right)}}.$$

Statuamus $v=\frac{f}{u}$, ut obtineamus $dt=\frac{ffd\varphi}{Euu}$ et

$$d\varphi=\frac{-Edu}{\sqrt{\left(Dff+4g(M+N)fu-EEuu+\frac{2g}{f}(cc-aa)(M+N)u^3\right)}}.$$

Ponamus $u=1+n\cos s$, ut ob $du=-nds\sin s$, differentiale du et propterea etiam dv duobus casibus evanescat: $s=0$ et $s=180^\circ$, ac necesse est, ut quoque denominator seu formula irrationalis evanescat iisdem casibus, quod fieri nequit, nisi ea factorem habeat $\sin s$. Facta autem substitutione $u=1+n\cos s$, quantitas signo radicali involuta abit in hanc formam, posito brevitatis causa $\frac{2g}{f}(cc-aa)(M+N)=L$

$$\begin{array}{llll} Dff & & & \\ +4g(M+N)f & +4ng(M+N)f\cos s & & \\ -EE & -2nEE\cos s & -nnEE\cos^2 s & \\ +L & +3nL\cos s & +3nnL\cos^2 s & +n^3L\cos^3 s \end{array}$$

Scribamus pro $\cos^2 s$ valorem $1-\sin^2 s$, ut sit $\cos^3 s=\cos s-\sin^2 s\cos s$, fietque haec quantitas

$$\begin{aligned} &+Dff+4g(M+N)f-(1+nn)EE+(1+3nn)L\\ &+(4ng(M+N)f-2nEE+n(3+nn)L)\cos s\\ &+nn(EE-3L-nL\cos s)\sin^2 s \end{aligned}$$

ac membra a $\sin^2 s$ immunia seorsim ad nihilum reducantur, ut constantes D et E per integrationem inductae per constantes novas assumtas f et n determinentur, quo pacto obtinebimus

$$EE=2fg(M+N)+\tfrac{1}{2}(3+nn)L$$

$$\text{et}\quad Dff+2(1-nn)fg(M+N)-\tfrac{1}{2}(1-nn)^2L=0,$$

$$\text{seu}\quad Dff=-2(1-nn)fg(M+N)+\tfrac{1}{2}(1-nn)^2L,$$

unde denominator irrationalis prodit

$$n\sin s\sqrt{(2fg(M+N)-\tfrac{1}{2}(3-nn)L-nL\cos s)}$$

hincque

$$d\varphi=\frac{Eds}{\sqrt{(2fg(M+N)-\frac{1}{2}(3-nn)L-nL\cos s)}}$$

$$\text{et}\quad dt=\frac{ffds}{(1+n\cos s)^2\sqrt{(2fg(M+N)-\frac{1}{2}(3-nn)L-nL\cos s)}},$$

unde haud difficulter quantitates φ et t per variabilem s, ex eaque etiam $v=\frac{f}{1+n\cos s}$ definire licet.

137. **Coroll. 1.** Si $n=0$, ob $v=f$ corpus N in circulo circa J revolvetur motu uniformi, quae celeritas angularis

$$\frac{d\varphi}{dt}=\frac{E}{ff}=\frac{1}{ff}\sqrt{\left(2fg\,(M+N)+\frac{3g}{f}(cc-aa)\,(M+N)\right)}$$

ubi pro L ejus valore, ex quo haec celeritas erit quoque

$$\frac{d\varphi}{dt}=\frac{\sqrt{2g\,(M+N)}}{f\sqrt{f}}\sqrt{\left(1+\frac{3\,(cc-aa)}{2ff}\right)}.$$

138. **Coroll. 2.** At si $n>1$, corpus N ita movebitur, ut absolutis angulis $s=0^0$, 360^0, 2.360^0, 3.360^0, etc. semper ad eandem distantiam minimam $v=\frac{f}{1+n}$ revertatur, angulis autem 180^0, 3.180^0, 5.180^0, etc. absolutis, ad distantiam maximam $v=\frac{f}{1-n}$ perveniat. Illis scilicet casibus in abside ima, his vero in abside summa versabitur.

139. **Coroll. 3.** Cum autem anguli s non sint angulis φ aequales, loca absidum non iisdem angulis $AJN=\varphi$ successive respondebunt, unde hoc motu linea absidum mobilis est censenda, nominis vero n, quo discrimen inter distantiam maximam et minimam definitur, haud incongrue eccentricitas, angulus s vero anomalia vera dicetur.

140. **Coroll. 4.** Si pro L valorem assumtum restituamus, erit

$$d\varphi=\frac{ds\sqrt{1+\frac{(3+nn)\,(cc-aa)}{2ff}}}{\sqrt{\left(1-\frac{(3-nn)\,(cc-aa)}{2ff}-\frac{n\,(cc-aa)\cos s}{ff}\right)}}$$

et

$$dt\sqrt{2fg\,(M+N)}=\frac{ffds}{(1+n\cos s)^2\sqrt{\left(1-\frac{(3-nn)\,(cc-aa)}{2ff}-\frac{n\,(cc-aa)\cos s}{ff}\right)}}.$$

Ab harum ergo duarum formularum integratione tota problematis solutio pendet.

141. **Scholion 1.** Fieri posse videtur, ut formulae hae irrationales adeo fiant imaginariae, id quod etiam in formula $\sqrt{\left(1+\frac{3\,(cc-aa)}{2ff}\right)}$ locum haberet, si esset $2ff+3cc<3aa$, quo casu vis attractrix in distantia f, quae est $=\frac{MN}{ff}\left(1+\frac{3\,(cc-aa)}{2ff}\right)$, fuerit negativa, quod utique est absurdum. Verum meminisse oportet formulas, quas supra pro vi attractrice invenimus, expressis verbis ex hac hypothesi esse deductas, quod distantia, quae hic est $=f$, sit praegrandis prae corporis attrahentis magnitudine, a qua litterae a et c pendent. Quare in omnibus his solutionibus hoc primarium est requisitum, ut quantitas f vehementer excedat a et c, hincque fractio $\frac{cc-aa}{ff}$ semper sit quam minima. Atque ob hanc causam formulae inventae semper realiter motum quaesitum pro nostro quidem instituto definire sunt censendae. Si enim motus ita esset comparatus, ut corpus N nimis ad alterum accederet, tum ne quidem ejus determinationem hic quidem suscipere liceret.

142. **Scholion 2.** Cum igitur quantitas $\frac{cc-aa}{ff}$ per hypothesin sit valde exigua, approximationibus adhibendis adipiscemur has formulas

$$d\varphi = ds\left(1 + \frac{3(cc-aa)}{2ff} + \frac{n(cc-aa)}{2ff}\cos s\right)$$

et $$dt\sqrt{2fg(M+N)} = \frac{ffds}{(1+n\cos s)^2}\left(1 + \frac{(3-nn)(cc-aa)}{4ff} + \frac{n(cc-aa)}{2ff}\cos s\right),$$

quarum illius integrale est

$$\varphi = \text{Const.} + \left(1 + \frac{3(cc-aa)}{2ff}\right)s + \frac{n(cc-aa)}{2ff}\sin s,$$

altera vero, prout excentricitas n fuerit unitate vel minor vel major vel eidem aequalis, singulari modo integrari debet, pro quo negotio ea in hanc formam transfundatur

$$dt\sqrt{2fg(M+N)} = \frac{ffds}{(1+n\cos s)^2}\left(1 + \frac{(1-nn)(cc-aa)}{4ff} + \frac{(cc-aa)ds}{2(1+n\cos s)}\right).$$

Jam vero casu $n < 1$ supra ostendimus esse

$$\int\frac{ds}{1+n\cos s} = \frac{1}{\sqrt{(1-nn)}}\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s}$$

et $$\int\frac{ds}{(1+n\cos s)^2} = \frac{1}{(1-nn)^{\frac{3}{2}}}\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} - \frac{n\sin s}{(1-nn)(1+n\cos s)},$$

deinde casu, quo $n = 1$,

$$\int\frac{ds}{1+\cos s} = \frac{\sin s}{1+\cos s} \quad \text{et} \quad \int\frac{ds}{(1+\cos s)^2} = \frac{(2+\cos s)\sin s}{3(1+\cos s)^2}.$$

Tum vero casu, quo $n > 1$,

$$\int\frac{ds}{1+n\cos s} = \frac{1}{\sqrt{(nn-1)}}\log\cdot\frac{n+\cos s+\sin s\sqrt{(nn-1)}}{1+n\cos s},$$

$$\int\frac{ds}{(1+n\cos s)^2} = \frac{n\sin s}{(nn-1)(1+n\cos s)} - \frac{1}{(nn-1)^{\frac{3}{2}}}\log\cdot\frac{n+\cos s+\sin s\sqrt{(nn-1)}}{1+n\cos s}.$$

Hinc ergo pro casu $n < 1$ habebimus

$$t\sqrt{2fg(M+N)} = \frac{ff}{(1-nn)^{\frac{3}{2}}}\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} - \frac{nff\sin s}{(1-nn)(1+n\cos s)}$$
$$+ \frac{3(cc-aa)}{4\sqrt{(1-nn)}}\text{Arc.}\cos\frac{n+\cos s}{1+n\cos s} - \frac{n(cc-aa)\sin s}{4(1+n\cos s)}.$$

Tum vero pro casu $n = 1$

$$t\sqrt{2fg(M+N)} = \frac{ff(2+\cos s)\sin s}{3(1+\cos s)^2} + \frac{(cc-aa)\sin s}{2(1+\cos s)}.$$

Ac denique pro casu $n > 1$

$$t\sqrt{2fg(M+N)} = \frac{nff\sin s}{(nn-1)(1+n\cos s)} - \frac{ff}{(nn-1)^{\frac{3}{2}}}\log\cdot\frac{n+\cos s+\sin s\sqrt{(nn-1)}}{1+n\cos s}$$
$$- \frac{n(cc-aa)\sin s}{4(1+n\cos s)} + \frac{3(cc-aa)}{4\sqrt{(nn-1)}}\log\cdot\frac{n+\cos s+\sin s\sqrt{(nn-1)}}{1+n\cos s}.$$

Hunc quidem casum, quo $n > 1$, quoniam in mundo nusquam locum habere videtur, relinquentes, alterum, quo $n < 1$ accuratius persequamur, et quo pacto motus commodissime definiri atque ad datum tempus assignari possit, videamus. Manifestum autem est hunc motum parum a motu in ellipsi facto, quem supra exposuimus, fore diversum.

143. **Problema.** Determinationem motus, quo corpus N in casu praecedentis problematis circa corpus M in plano aequatoris revolvitur, ad calculum revocare.

Solutio. Primo cum s exprimat anomaliam veram corporis N, hoc est ejus longitudinem ab abside ima computatam, littera vero φ longitudinem veram denotet a directione quapiam fixa computatam, ab eadem hac directione fixa longitudo absidis imae erit $= \varphi - s$, quae ergo ita definitur, ut sit

$$\varphi - s = \text{Const.} + \frac{(cc - aa)(3s + n \sin s)}{2ff},$$

unde patet lineam absidum non quiescere, sed in consequentia proferri, si sit $cc > aa$; sin autem fuerit $cc < aa$, retro moveri. Corpus scilicet N ab abside ima egressum ad absidem summam appellet confecto angulo $\varphi = \left(1 + \frac{3(cc - aa)}{2ff}\right) 180^0$, hoc est majori quam 180^0 si $cc > aa$; contra autem minori si $cc < aa$. In genere autem inventa anomalia vera $= s$, erit longitudo

$$\varphi = \text{Const.} + \left(1 + \frac{3(cc - aa)}{2ff}\right) s + \frac{n(cc - aa)}{2ff} \sin s.$$

Sin autem anomaliam veram s spectemus ut datam, erit distantia $JN = v = \frac{f}{1 + n \cos s}$, ubi f contemplamur ut semiparametrum orbitae, et n ejus excentricitatem, etiamsi orbita non sit elliptica. Tum vero pro relatione inter tempus t et anomaliam veram s commode exprimenda introducatur anomalia excentrica σ, ita ut sit

$$\cos \sigma = \frac{n + \cos s}{1 + n \cos s} \quad \text{et} \quad \sin \sigma = \frac{\sin s \sqrt{(1 - nn)}}{1 + n \cos s},$$

unde vicissim ex data σ fit

$$\cos s = \frac{\cos \sigma - n}{1 - n \cos \sigma} \quad \text{et} \quad \sin s = \frac{\sin \sigma \sqrt{(1 - nn)}}{1 - n \cos \sigma}.$$

His positis habebimus

$$t \sqrt{2fg(M + N)} = \text{Const.} + \frac{ff(\sigma - n \sin \sigma)}{(1 - nn)\sqrt{(1 - nn)}} + \frac{(cc - aa)(3\sigma - n \sin \sigma)}{4\sqrt{(1 - nn)}},$$

unde vicissim pro dato tempore t primo anomalia excentrica σ, ex hacque porro vera s, hincque tam longitudo φ quam distantia $JN = v$ definiri poterit.

144. **Coroll. 1.** Cum fractio $\frac{cc - aa}{ff}$ sit quam minima, motus lineae absidum erit tardissimus, atque singulis revolutionibus corporis N tantum per angulum $\frac{3(cc - aa)}{2ff} \cdot 360^0 = \frac{cc - aa}{ff} \cdot 540^0$ progredietur.

145. **Coroll. 2.** Tempus porro integrae revolutionis, quo corpus ab abside vel ima vel summa egressum iterum ad eandem revertitur, confecto angulo $\varphi = \left(1 + \frac{3(cc - aa)}{2ff}\right) 360^0$, reperitur

ponendo $s = 360^0$, unde fit quoque $\sigma = 360^0 = 2\pi$. Ex quo tempus unius revolutionis erit

$$= \frac{2\pi ff}{(1-nn)^{\frac{3}{2}}\sqrt{2fg(M+N)}} + \frac{3\pi(cc-aa)}{2(1-nn)^{\frac{1}{2}}\sqrt{2fg(M+N)}},$$

vel posito semiaxe transverso $\frac{f}{1-nn} = k$, fiet hoc tempus $= \frac{\pi}{\sqrt{2g(M+N)}}\left(2k\sqrt{k} + \frac{3(cc-aa)}{2(1-nn)\sqrt{k}}\right)$.

146. **Coroll. 3.** Si anomaliam mediam seu angulum tempori proportionalem, qui absoluta una revolutione evadat $= 360^0$, introducamus, eamque ponamus $= \tau$, debet esse

$$t\sqrt{2fg(M+N)} = \tau\left(\frac{ff}{(1-nn)\sqrt{(1-nn)}} + \frac{3(cc-aa)}{4\sqrt{(1-nn)}}\right),$$

hincque fiet
$$\tau = \sigma - n\left(1 - \frac{(1-nn)(cc-aa)}{2ff}\right)\sin\sigma,$$

unde, ut supra, facile ex data anomalia media τ anomalia excentrica σ colligitur.

147. **Scholion 1.** Ut intelligamus quanta hujusmodi perturbatio ob figuram corporum coelestium non sphaericam oriri debeat, tribuamus corpori M, quod ex materia constet homogenea, figuram sphaeroidis elliptici, revolutione ellipsis circa axem CJ geniti, cujus alter semiaxis JC, circa quem fit revolutio, sit $= C$, alter vero seu semidiameter aequatoris $= A$, ita ut hoc sphaeroides sit compressum si $A > C$, elongatum vero si $A < C$. Jam supra § 44 vidimus fore pro nostris momentis inertiae $aa = bb = \frac{1}{5}(AA + CC)$ et $cc = \frac{2}{5}AA$, unde fit $cc - aa = \frac{1}{5}(AA - CC)$. Corpus ergo sphaeroidicum compressum, cujusmodi est sol et terra ac sine dubio omnes planetae principales, efficit, ut lineae absidum progrediantur, quae regrederentur, si sphaeroides esset oblongum. Jam pro terra est quasi $A = \frac{201}{200}C$ et $AA - CC = \frac{1}{100}CC$, ergo $cc - aa = \frac{1}{500}CC$. Hinc si statuatur $f = 60C$, ut fere evenit in luna, erit $\frac{cc-aa}{ff} = \frac{1}{500.60.60}$. Quare hinc linea absidum orbitae lunaris singulis revolutionibus seu mensibus progrederetur per $\frac{1}{500.60.60}\cdot 540^0 = 1''\,5'''$, et unius anni intervallo per $14''$, qui effectus prae eo, qui ab aliis causis oritur, facile negligi potest. In sole autem $cc - aa$ multo minus est quam $\frac{1}{500}CC$, unde in planetis primariis hinc nulla perturbatio sensibilis oriri est censenda. In Jove autem, quia ob tam celerem motum vertiginis est circiter $A = \frac{11}{10}C$, erit $AA - CC = \frac{1}{5}CC$ hincque $cc - aa = \frac{1}{25}CC$. Cum jam pro primo satellite sit quasi $f = 6C$, una revolutione hujus satellitis linea absidum progreditur per angulum $\frac{1}{25.6.6}\cdot 540^0 = 36'$, siquidem orbitam ejus in plano aequatoris Jovis statuamus, unde cum hic effectus producatur tempore 42 horarum, intervallo unius diei erit $20'$ et unius anni $121^0\,45'$, cujusmodi velox absidum motus nusquam alibi est observatus; in reliquis autem Jovis satellitibus multo minor esse debet, ob majorem eorum distantiam. At Saturnus, si annulum cum eo conjunctim spectemus, referet sphaeroides multo magis compressum, ex quo in ejus satellitibus multo velocior motus absidum generari debet. Hic certe maxime notatu est dignum in orbitis satellitum Jovis, ac praecipue primi, tam enormem

apsidum mutabilitatem inesse debere, quae tantum a figura Jovis non sphaerica proficiscatur; quod phaenomenon si per observationes confirmari posset, mirifice theoriae attractionis universalitatem confirmaret.

148. **Scholion 2.** Quanquam in hoc motu, quem hic definivimus, tam via a corpore descripta quam temporis ratio areis proportionalis maxime est transcendens, tamen calculum ita commode administrare licuit, ut determinatio motus vix difficilior, quam in casu ellipsis simplicis evaderet. Totum scilicet discrimen huc est perductum, ut linea absidum mobilis statueretur, dum reliqua omnia prorsus cum motu elliptico supra exposito conveniunt. Hoc compendio Astronomi jam pridem feliciter sunt usi, dum motus planetarum primariorum ita repraesentant, ac si in ellipsibus mobilibus circa solem revolverentur, in motu autem lunae insuper tam excentricitatem quam parametrum ellipsis variabilem statui oportere agnoverunt; quae idea eximium calculi alias intricatissimi compendium largitur. Atque non solum haec ita se habent, quando curva percursa sita est in eodem plano, sed etiam quando ejus planum est variabile; tum autem hujus variabilitatis rationem singulari modo ita ad quodpiam planum fixum referri convenit, ut ad quodvis tempus tam intersectio quam inclinatio planorum definiatur.

149. **Scholion 3.** Hoc modo approximationem institui conveniet, quando formulae analyticae, quibus motus determinatur, resolutionem non admittunt, quemadmodum in hoc capite usu venit, ubi hunc solum postremum casum, quo corpus M bina momenta inertiae respectu axium JA et JB aequalia habere, alterumque corpus N in ipso horum axium plano AJB moveri ponebatur, expedire licuit. Fundamentum autem hujus approximationis in hoc est situm, quod inter vires corpus N sollicitantes una prae ceteris eminet, quae ad punctum quasi fixum J dirigitur et quadratis distantiarum reciproce est proportionalis, reliquae autem vires prae hac sint valde exiguae. Tum enim motus corporis N non multum a ratione motus in sectione conica facti differet, cujus aberrationem tantum ab ista lege definivisse sufficiet. Quemadmodum ergo his casibus ope approximationis ad solutionem pervenire liceat, in sequente capite generatim explicabimus, in quo duplex investigatio erit instituenda, prout motus corporis N vel in eodem plano absolvetur, vel secus.

Caput V.

Determinatio motus corporis, quando inter vires, quibus sollicitatur, una ad punctum fixum tendens, quadrato distantiae ab eo est reciproce proportionalis, reliquae vero vires prae illa sunt valde parvae.

150. **Problema.** (Fig. 182.) Si corpus N circa punctum quasi fixum J in eodem plano moveatur, atque ad id trahatur vi quadrato distantiae reciproce proportionali, praeterea vero a viribus quibuscunque illius respectu valde parvis, corporis motum definire.

Solutio. Elapso tempore T sit distantia $JN = \varrho$ et angulus $AJN = \varphi$, ut sint

$$JX = x = \varrho \cos \varphi \quad \text{et} \quad XN = y = \varrho \sin \varphi.$$

Ponamus jam vim secundum directionem NJ esse $=\frac{LN}{vv}$, ac praeterea adesse vires valde parvas P et NQ, secundum directiones JX et XN agentes, et habebimus has aequationes:

$$ddx = -2g\,dt^2\left(\frac{Lx}{v^3} + P\right) \quad \text{et} \quad ddy = -2g\,dt^2\left(\frac{Ly}{v^3} + Q\right),$$

unde concludimus: $xddy - yddx = -2g\,dt^2\,(Qx - Py)$, hincque integrando

$$xdy - ydx = -2g\,dt\int dt\,(Qx - Py), \quad \text{seu} \quad vv\,d\varphi = -2g\,dt\int v\,dt\,(Q\cos\varphi - P\sin\varphi).$$

Statuamus nunc $v = \frac{p}{1+q\cos s}$, ubi non solum angulus s, qui denotet anomaliam veram, sed etiam semiparameter p et excentricitas q sint quantitates variabiles, quarum variabilitas autem sit valde parva utpote a viribus P et Q proficiscens, quae si evanescerent, utique tam p quam q forent quantitates constantes. Ponamus brevitatis gratia $S = -2g\int v\,dt\,(Q\cos\varphi - P\sin\varphi)$, ut habeamus

$$d\varphi = \frac{S\,dt\,(1+q\cos s)^2}{pp}.$$

Deinde ex primis aequationibus concludimus

$$xddx + yddy = -2g\,dt^2\left(\frac{L}{v} + Px + Qy\right);$$

at est $\quad xddx + yddy + dx^2 + dy^2 = d.vdv = vddv + dv^2$ et $dx^2 + dy^2 = dv^2 + vv\,d\varphi^2$,

hincque
$$xddx + yddy = vddv - vv\,d\varphi^2,$$

$$\text{seu} \quad ddv - v\,d\varphi^2 = -2g\,dt^2\left(\frac{L}{vv} + P\cos\varphi + Q\sin\varphi\right),$$

ubi si pro $d\varphi$ valorem inventum substituamus, nanciscemur

$$\frac{ddv}{dt} = \frac{SS\,dt\,(1+q\cos s)^3}{p^3} - \frac{2gL\,dt\,(1+q\cos s)^2}{pp} - 2g\,dt\,(P\cos\varphi + Q\sin\varphi).$$

Hic primum observo si praeter P et Q etiam excentricitas q evanesceret, prodire debere $ddv = 0$ unde necesse est sit $SS = 2gLp$ et $S = \sqrt{2gLp}$. Quare habebimus

$$dS = \frac{dp}{2\sqrt{p}}\sqrt{2gL} = -2gv\,dt\,(Q\cos\varphi - P\sin\varphi),$$

ideoque
$$dp = \frac{-4g\,dt\,(Q\cos\varphi - P\sin\varphi)\,p\sqrt{p}}{(1+q\cos s)\sqrt{2gL}} \quad \text{et} \quad d\varphi = \frac{dt\,(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}}.$$

Tum vero nostra aequatio adhuc resolvenda erit

$$\frac{ddv}{dt} = \frac{2gLq\,dt\cos s}{pp}(1+q\cos s)^2 - 2g\,dt\,(P\cos\varphi + Q\sin\varphi).$$

Jam quia per hypothesin dum fit $\sin s = 0$, etiam $\frac{dv}{dt}$ evanescere debet, statuamus $\frac{dv}{dt} = Vq\sin s$ eritque

$$Vq\,dt\sin s = dv = \frac{-4g\,dt\,(Q\cos\varphi - P\sin\varphi)\,p\sqrt{p}}{(1+q\cos s)^2\sqrt{2gL}} - \frac{pd.\,q\cos s}{(1+q\cos s)^2},$$

ita ut sit

$$d.q\cos s = \frac{-Vqdt\sin s\,(1+q\cos s)^2}{p} - \frac{4gdt\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{\sqrt{2gL}}.$$

Porro autem ob $\frac{dv}{dt} = qdV\sin s + Vd.q\sin s$, erit

$$d.q\sin s = \frac{-qdV\sin s}{V} + \frac{2gLqdt\cos s}{Vpp}(1+q\cos s)^2 - \frac{2gdt\,(P\cos\varphi + Q\sin\varphi)}{V},$$

ex quibus duabus aequationibus concluditur

$$dq = \frac{-Vqdt\sin s\cos s\,(1+q\cos s)^2}{p} - \frac{4gdt\cos s\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{\sqrt{2gL}}$$
$$-\frac{qdV\sin^2 s}{V} + \frac{2gLqdt\sin s\cos s}{Vpp}(1+q\cos s)^2 - \frac{2gdt\sin s\,(P\cos\varphi + Q\sin\varphi)}{V},$$

quae expressio evanescere debet casu $P=0$ et $Q=0$, ubi simul V fieret constans, ex qua conditione prodit

$$VV = \frac{2gL}{p} \quad \text{et} \quad V = \frac{\sqrt{2gL}}{\sqrt{p}} \quad \text{et} \quad dv = qdt\sin s\sqrt{\frac{2gL}{p}}.$$

Praeterea ob

$$\frac{dV}{V} = \frac{-dp}{2p} = \frac{2gdt\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{(1+q\cos s)\sqrt{2gL}},$$

$$d.q\cos s = \frac{-qdt\sin s\,(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}} - \frac{4gdt\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{\sqrt{2gL}}$$

$$d.q\sin s = \frac{qdt\cos s\,(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}} - \frac{2gdt\,(P\cos\varphi + Q\sin\varphi)\sqrt{p}}{\sqrt{2gL}}$$
$$- \frac{2gqdt\sin s\,(Q\cos\varphi - P\sin\varphi)\sqrt{p}}{(1+q\cos s)\sqrt{2gL}},$$

unde colligimus

$$dq = \frac{-2gdt\sqrt{p}}{\sqrt{2gL}}\left(2(Q\cos\varphi - P\sin\varphi)\cos s + (P\cos\varphi + Q\sin\varphi)\sin s + \frac{q(Q\cos\varphi - P\sin\varphi)\sin^2 s}{1+q\cos s}\right),$$

$$qds = \frac{qdt(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}} + \frac{2gdt\sqrt{p}}{\sqrt{2gL}}\left(2(Q\cos\varphi - P\sin\varphi)\sin s - (P\cos\varphi + Q\sin\varphi)\cos s - \frac{q(Q\cos\varphi - P\sin\varphi)\sin s\cos s}{1+q\cos s}\right),$$

ita ut hinc sit

$$ds = \frac{dt(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}} + \frac{2gdt\sqrt{p}}{\sqrt{2gL}}\left(\frac{2(Q\cos\varphi - P\sin\varphi)\sin s}{q} - \frac{(P\cos\varphi + Q\sin\varphi)\cos s}{q} - \frac{(Q\cos\varphi - P\sin\varphi)\sin s\cos s}{1+q\cos s}\right).$$

Inde autem variatio excentricitatis q definitur, aeque ac semiparametri p, quibus inventis pro ipso motu erit

$$v = \frac{p}{1+q\cos s} \quad \text{et} \quad d\varphi = \frac{dt\,(1+q\cos s)^2\sqrt{2gL}}{p\sqrt{p}}.$$

Cum deinde $\varphi - s$ designet longitudinem absidis imae, et haec erit variabilis, habebiturque

$$d(\varphi - s) = \frac{2gdt\sqrt{p}}{q\sqrt{2gL}}\left((P\cos\varphi + Q\sin\varphi)\cos s - 2(Q\cos\varphi - P\sin\varphi)\sin s + \frac{q(Q\cos\varphi - P\sin\varphi)\sin s\cos s}{1+q\cos s}\right)$$

sicque omnia, quae ad motus determinationem attinent, sunt determinata.

151. **Coroll. 1.** Si ponamus $Q\cos\varphi - P\sin\varphi = T$ et $Q\sin\varphi + P\cos\varphi = U$, ut aequationes resolvendae sint:

$$v dd\varphi + 2 dv d\varphi = -2gTdt^2 \quad \text{et} \quad ddv - v d\varphi^2 = \frac{-2gLdt^2}{vv} - 2gUdt^2,$$

eae posito $v = \frac{p}{1+q\cos s}$ ita resolventur, ut sit

$$1.\ d\varphi = \frac{dt\,(1+q\cos s)^2}{p\sqrt{p}}\sqrt{2gL}, \qquad 2.\ d\varphi - ds = \frac{2gdt\sqrt{p}}{q\sqrt{2gL}}\left(U\cos s - 2T\sin s + \frac{qT\sin s\cos s}{1+q\cos s}\right),$$

$$3.\ dp = \frac{-4gTpdt\sqrt{p}}{(1+q\cos s)\sqrt{2gL}}, \qquad 4.\ dq = \frac{-2gdt\sqrt{p}}{\sqrt{2gL}}\left(2T\cos s + U\sin s + \frac{qT\sin^2 s}{1+q\cos s}\right).$$

152. **Coroll. 2.** Si ex formulis N° 2. 3. 4. quantitates T et U elidantur, pervenietur ad hanc aequationem:

$$\frac{dp}{p} = \frac{dq\cos s + q\,(d\varphi - ds)\sin s}{1+q\cos s},$$

quae integrata quatenus licet dat

$$l\,\frac{p}{1+q\cos s} = \int\frac{qd\varphi\sin s}{1+q\cos s} = \int\frac{qdt\,(1+q\cos s)\sin s}{p\sqrt{p}}\sqrt{2g}\,L.$$

153. **Coroll. 3.** Cum quantitates P et Q sint per hypothesin valde parvae, erunt quantitates p et q fere constantes et $d\varphi = ds$, unde fit

$$dt = \frac{pds\sqrt{p}}{(1+q\cos s)^2\sqrt{2gL}},$$

cujus integrale spectatis p et q ut constantibus exhiberi poterit, quod cum sit prope verum, sufficiet deinceps hunc valorem pro dt in formulis 2, 3, 4 posuisse, ex iisque sumta sola s pro variabili valores proxime veros pro $\varphi - s$, p et q elicuisse.

154. **Coroll. 4.** Hoc autem pro dt valore inducto, aequationes nostrae evolvendae erunt:

$$2\quad d\varphi - ds = \frac{ppds}{Lq\,(1+q\cos s)^2}\left(U\cos s - 2T\sin s + \frac{qT\sin s\cos s}{1+q\cos s}\right),$$

$$3.\quad dp = \frac{-2Tp^3 ds}{L\,(1+q\cos s)^3},$$

$$4.\quad dq = \frac{-ppds}{L\,(1+q\cos s)^2}\left(2T\cos s + U\sin s + \frac{qT\sin^2 s}{1+q\cos s}\right).$$

Revera autem in his formulis pro ds scribi oporteret $d\varphi$, sed quia saltem proxime est $d\varphi = ds$, ... in appropinquatione uti licebit.

155. **Scholion I.** Hoc modo solutio problematis ad determinationem motus in ellipsi variabili perducitur, ita ut ratio motus similis sit illi, quam supra pro casu duorum corporum sphaericorum assignavimus, praeterquam quod hic elementa ellipsis omnia variabilia statuantur. Primo enim tam semiparameter ellipsis p quam excentricitas q est variabilis, tum vero etiam ipsa linea absidum mobilis assumitur, denotante angulo s anomaliam veram, secundum eandem ideam, quam supra constituimus. Atque haec reductio eo magis est notatu digna, quod quaedam operationes prorsus

nostro sint institutae, ex quo apparet infinitis aliis modis etiam posito $\varrho = \frac{p}{1+q\cos s}$ relationem inter differentialia dp, dq, ds et $d\varphi$ ita constitui posse, ut motus rationi satisfiat. Loco enim determinationum $SS = 2gLp$ et $VV = \frac{2gL}{p}$, eosdem valores quantitatibus quibusdam exiguis per vires T et V definiendis augere liceret, quo pacto conditiones propositae aeque impleri possent, ut scilicet tam $q = 0$ quam $\sin s = 0$, evanescat $\frac{d\varrho}{dt}$, insuperque casu $T = 0$, $V = 0$ et $q = 0$ prodeat $V = 0$. Cum enim loco unius variabilis ϱ tres novae p, q et s introducantur, mirum non est binarum determinationem arbitrio nostro relinqui, quam ita constitui convenit, ut calculus commodissimus reddatur, in quo quidem negotio saepenumero maxima difficultas deprehenditur. Atque in solutione quidem, qua hic sumus usi, parum congruere videtur, quod expressio pro $d\varphi - ds$ inventa per excentricitatem q sit divisa, qua conditione determinatio motus lineae absidum lubrica redditur, praecipue quando excentricitas q est valde parva. Siquidem calculum perfecte expedire liceret, nullum incommodum hinc esset metuendum, quoniam perpetuo absides ibi existunt, ubi distantia ϱ est vel maxima vel minima, ita ut hic nulli incertitudini locus relinquatur. At cum approximatione contenti esse debeamus, ob hanc causam haud levia impedimenta occurrere possunt.

156. **Scholion 2.** Solutioni igitur summam extensionem tribuamus, et cum aequationes propositae sint:

$$\varrho dd\varphi + 2d\varrho d\varphi = -2gTdt^2, \qquad dd\varrho - \varrho d\varphi^2 = \frac{-2gLdt^2}{\varrho\varrho} - 2gVdt^2,$$

posito $\varrho = \frac{p}{1+q\cos s}$, statuamus $-2g\int T\varrho dt = \sqrt{2gp}\,(L+X) = \frac{\varrho\varrho\, d\varphi}{dt}$, eritque

$$\frac{-2gTpdt}{1+q\cos s} = \frac{dp}{2\sqrt{p}}\sqrt{2g}\,(L+X) + \frac{dX\sqrt{2gp}}{2\sqrt{(L+X)}}, \quad \text{hincque}$$

$$dp = \frac{-2Tpdt\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}} - \frac{pdX}{L+X} \quad \text{et} \quad d\varphi = \frac{dt\,(1+q\cos s)^2\sqrt{2gp}\,(L+X)}{pp}.$$

Porro statuatur $\frac{d\varrho}{dt} = q\sin s\sqrt{\frac{2g\,(L+Y)}{p}}$, eritque primo

$$qdt\sin s\sqrt{\frac{2g\,(L+Y)}{p}} = \frac{-2Tpdt\sqrt{2gp}}{(1+q\cos s)^2\sqrt{(L+X)}} - \frac{pdX}{(1+q\cos s)(L+X)} - \frac{pd.q\cos s}{(1+q\cos s)^2},$$

unde colligimus

$$d.q\cos s = \frac{-qdt\sin s\,(1+q\cos s)^2\sqrt{2g\,(L+Y)}}{p\sqrt{p}} - \frac{2Tdt\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{(1+q\cos s)\,dX}{L+X}.$$

Deinde ex forma $\frac{d\varrho}{dt}$ assumta deducimus

$$\frac{dd\varrho}{dt} = \frac{\sqrt{2g\,(L+Y)}}{\sqrt{p}}\,d.q\sin s - \frac{qdp\sin s\sqrt{2g\,(L+Y)}}{2p\sqrt{p}} + \frac{qdY\sin s\sqrt{2g}}{2\sqrt{p\,(L+Y)}}, \quad \text{seu}$$

$$\frac{dd\varrho}{dt} = \frac{\sqrt{2g\,(L+Y)}}{\sqrt{p}}\,d.q\sin s + \frac{2gTqdt\sin s\sqrt{(L+Y)}}{(1+q\cos s)\sqrt{(L+X)}} + \frac{qdX\sin s\sqrt{2g\,(L+Y)}}{2\,(L+X)\sqrt{p}} + \frac{qdY\sin s\sqrt{2g}}{2\sqrt{p\,(L+Y)}}.$$

Ex aequatione proposita est

$$\frac{ddv}{dt} = \frac{2gdt\,(1+q\cos s)^3\,(L+X)}{pp} - \frac{2gLdt\,(1+q\cos s)^2}{pp} - 2gVdt, \quad \text{seu}$$

$$\frac{ddv}{dt} = \frac{2gLqdt\cos s\,(1+q\cos s)^2}{pp} + \frac{2gXdt\,(1+q\cos s)^3}{pp} - 2gVdt,$$

qua expressione cum praecedente collata fit

$$d.q\sin s = \frac{2gLqdt\cos s\,(1+q\cos s)^2}{p\sqrt{2gp}\,(L+Y)} + \frac{2gXdt\,(1+q\cos s)^3}{p\sqrt{2gp}\,(L+Y)} - \frac{2gVdt\sqrt{p}}{\sqrt{2g}\,(L+Y)} - \frac{Tqdt\sin s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}}$$
$$- \frac{qdX\sin s}{2(L+X)} - \frac{qdY\sin s}{2(L+Y)}.$$

Hinc concludimus fore

$$dq = \frac{2gXdt\sin s\,(1+q\cos s)^3}{p\sqrt{2gp}\,(L+Y)} - \frac{2gVdt\sin s\sqrt{p}}{\sqrt{2g}\,(L+Y)} - \frac{dX\cos s\,(1+q\cos s)}{L+X} - \frac{qdX\sin^2 s}{2(L+X)}$$
$$- \frac{2gYqdt\sin s\cos s(1+q\cos s)^2}{p\sqrt{2gp}\,(L+Y)} - \frac{2Tdt\cos s\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{Tqdt\sin^2 s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}} - \frac{qdY\sin^2 s}{2(L+Y)},$$

$$qds = \frac{2gLqdt\,(1+q\cos s)^2}{p\sqrt{2gp}\,(L+Y)} + \frac{2gYqdt\sin^2 s\,(1+q\cos s)^3}{p\sqrt{2gp}\,(L+Y)} + \frac{2Tdt\sin s\sqrt{2gp}}{\sqrt{(L+X)}} + \frac{dX\sin s\,(1+q\cos s)}{L+X}$$
$$+ \frac{2gXdt\cos s\,(1+q\cos s)^3}{p\sqrt{2gp}\,(L+Y)} - \frac{Vdt\cos s\sqrt{2gp}}{\sqrt{(L+Y)}} - \frac{qdX\sin s\cos s}{2(L+X)}$$
$$- \frac{Tqdt\sin s\cos s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}} - \frac{qdY\sin s\cos s}{2(L+Y)}.$$

Si jam quantitates arbitrariae X et Y ita accipi possent, ut haec postrema expressio per q fieret divisibilis, incommodum supra memoratum tolleretur, id quod eveniret, si fieret

$$\frac{2gXdt\cos s}{p\sqrt{2gp}\,(L+Y)} + \frac{2Tdt\sin s\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{Vdt\cos s\sqrt{2gp}}{\sqrt{(L+Y)}} + \frac{dX\sin s}{L+X} = 0,$$

vel formulae per q multiplicatae.

En ergo has determinationes, quae ob binas arbitrarias X et Y, maxime generales sunt habendae

1. $\quad d\varphi = \dfrac{dt\,(1+q\cos s)^2\sqrt{2g\,(L+X)}}{p\sqrt{p}}$ existente $v = \dfrac{p}{1+q\cos s}$,

2. $$d\varphi - ds = \frac{dt\,(1+q\cos s)^2\sqrt{2g}}{p\sqrt{p}\,(L+Y)}\left(\sqrt{(L+X)}\,(L+Y) - L - Y\sin^2 s - \frac{1}{q}X\cos s\,(1+q\cos s)\right)$$
$$- \frac{2Tdt\sin s\sqrt{2gp}}{q\sqrt{(L+X)}} + \frac{Vdt\cos s\sqrt{2gp}}{q\sqrt{(L+Y)}} + \frac{Tdt\sin s\cos s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}}$$
$$+ \frac{dX\sin s\cos s}{2(L+X)} + \frac{dY\sin s\cos s}{2(L+Y)} - \frac{dX\sin s\,(1+q\cos s)}{q(L+X)},$$

3. $$dp = \frac{-2Tpdt\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}} - \frac{pdX}{L+X},$$

4. $$dq = \frac{dt\sin s\,(1+q\cos s)^2\sqrt{2g}}{p\sqrt{p}\,(L+Y)}\Big(X(1+q\cos s) - Yq\cos s\Big)$$
$$- \frac{2Tdt\cos s\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{Vdt\sin s\sqrt{2gp}}{\sqrt{(L+Y)}} - \frac{Tqdt\sin^2 s\sqrt{2gp}}{(1+q\cos s)\sqrt{(L+X)}}$$
$$- \frac{qdX\sin^2 s}{2(L+X)} - \frac{qdY\sin^2 s}{2(L+Y)} - \frac{dX\cos s\,(1+q\cos s)}{L+X}.$$

...autem est quantitates X et Y valde parvas capi debere, easque quatenus a p et q pen-... pro constantibus esse habendas; sin autem insuper angulum φ vel s involvant, in earum diffe-... loco $d\varphi$ vel ds scribi posse

$$\frac{dt\,(1+q\cos s)^2\sqrt{2g\,(L+X)}}{p\sqrt{p}}.$$

Denique meminisse juvabit esse $dv = qdt\sin s\sqrt{\frac{2g\,(L+Y)}{p}}$.

... **Scholion 3.** Ut pro litteris X et Y quovis casu commodissimi valores eligantur, id ... videtur, ut quantitatum p et q variabilitas tam exigua reddatur quam fieri potest. Quodsi ... fieri queat, ut hae duae quantitates p et q evadant constantes, nullum est dubium, quin tum ... simplicissimo modo repraesentetur. Semper quidem has litteras X et Y ita definire liceret, ut fieret tam $dp = 0$ quam $dq = 0$, verum tum plerumque reliquae formulae nimis prodirent complicatae, quam ut hinc ullum commodum consequeremur; quare in hoc negotio ita versari conveniet, ut si non commode formulae pro dp et dq inventae ad nihilum redigi queant, eae saltem tam parvae ... quam fieri poterit, neque tamen ad hoc valores nimis perplexi pro X et Y adhibeantur: ... imprimis cavendum est, ne hi valores unquam limites quantitatum prae L valde exiguarum superent. Quo igitur hoc judicium ratione formulae dq facilius instituatur, plerumque conveniet eam ... transformari, ut quantitas v cum suo differentiali dv, ponendo

$$1+q\cos s = \frac{p}{v} \quad \text{et} \quad qdt\sin s = \frac{dv\sqrt{p}}{\sqrt{2g\,(L+Y)}},$$

introducatur. Hoc modo obtinebimus

$$dq = \frac{pdv}{qvv\,(L+Y)}\left(\frac{pX}{v} - \frac{pY}{v} + Y\right) - \frac{2\,Tdt\cos s\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{pdX\cos s}{v\,(L+X)} - \frac{Vpdv}{q\,(L+Y)} - \frac{Tvdv\sin s}{\sqrt{(L+X)}\,(L+Y)}$$
$$- \frac{qdX\sin^2 s}{2\,(L+X)} - \frac{qdY\sin^2 s}{2\,(L+Y)}.$$

Cum autem sit

$$dp = -\frac{2\,Tvdt\sqrt{2gp}}{\sqrt{(L+X)}} - \frac{pdX}{L+X},$$

si hinc jam valorem idoneum pro X elegerimus, habebimus

$$dq = \frac{pdv}{qv^3\,(L+Y)}\,(pX - pY + vY) + \frac{dp\cos s}{v} - \frac{Vpdv}{q\,(L+Y)} - \frac{Tvdv\sin s}{\sqrt{(L+X)}\,(L+Y)} + \frac{qdp\sin^2 s}{2p}$$
$$+ \frac{Tqvdt\sin^2 s\sqrt{2gp}}{p\sqrt{(L+X)}} - \frac{qdY\sin^2 s}{2\,(L+Y)},$$

ubi termini littera T affecti se mutuo destruunt. Multiplicemus per q, et ob $q\cos s = \frac{p}{v} - 1$ et $qq\sin^2 s = qq - \left(\frac{p}{v} - 1\right)^2$, habebimus

$$qdq = \frac{pdv}{v^3\,(L+Y)}\,(pX - pY + vY) + \frac{dp\,(p-v)}{vv} - \frac{Vpdv}{L+Y} + \frac{qqdp}{2p} - \frac{dp\,(p-v)^2}{2pvv} - \frac{dY\,(qqvv - (p-v)^2)}{2vv\,(L+Y)},$$

quae reducitur ad hanc formam commodiorem

$$qdq = \frac{pdv\,(pX - pY + vY - Yv^3)}{v^3(L+Y)} - \frac{dp\,(1-qq)}{2p} - \frac{dY\,(qqvv - (p-v)^2)}{2vv\,(L+Y)},$$

unde quovis casu haud difficulter maxime idoneus valor pro Y assumendus colligitur. Velimus $\frac{p}{1-qq} = r$, fiet

$$\frac{dr}{r} = \frac{2rdv\,(pX - pY + vY - Yv^3) + vdY\,(pr - 2rv + vv)}{v^3\,(L+Y)},$$

quae formula si ad nihilum redigi possit, commodissimam solutionem suppeditabit. Videamus ergo quantum fructum hinc colligere queamus pro casu praecedentis capitis, ubi corpus N circa alterum M in plano AJB movetur.

158. **Problema.** Si corpus sphaericum N circa corpus M, figura quacunque praeditum, quod omni motu gyratorio destitutum ponitur, ita moveatur, ut perpetuo in plano binorum axium principalium AJB maneat, ejus motum definire.

Solutio. Maneant omnia ut in problemate § 128, ac tantum opus est, ut hic ponamus $z=0$, unde fiet $x = v\cos\varphi$ et $y = v\sin\varphi$. Quod si jam illas formulas ad has, quibus hic utimur, accommodemus, habebimus $L = M + N$ et

$$P = \frac{3L\cos\varphi}{2v^4}\,(3aa + bb + cc - 5aa\cos^2\varphi - 5bb\sin^2\varphi),$$

$$Q = \frac{3L\sin\varphi}{2v^4}\,(aa + 3bb + cc - 5aa\cos^2\varphi - 5bb\sin^2\varphi),$$

unde deducimus

$$T = \frac{3L\,(bb-aa)\sin\varphi\cos\varphi}{v^4} = \frac{3L\,(bb-aa)\sin 2\varphi}{2v^4},$$

$$V = \frac{3L}{4v^4}\,(2cc - aa - bb + 3\,(bb-aa)\cos 2\varphi).$$

Statuamus brevitatis gratia $bb - aa = n$ et $2cc - aa - bb = 2m$, eritque

$$T = \frac{3nL\sin 2\varphi}{2v^4} \quad \text{et} \quad V = \frac{3mL}{2v^4} + \frac{9nL\cos 2\varphi}{4v^4}.$$

Ponatur nunc $v = \frac{p}{1+q\cos s}$, et cum invenerimus

$$dp = \frac{-3nLdt\sin 2\varphi\,\sqrt{2gp}}{v^3\sqrt{(L+X)}} - \frac{pdX}{L+X},$$

notetur esse $d\varphi = \frac{dt\sqrt{2gp\,(L+X)}}{vv}$, unde fit

$$dp = \frac{-3nLd\varphi\sin 2\varphi}{v\,(L+X)} - \frac{pdX}{L+X},$$

ad quem valorem diminuendum ponamus

$$X = \frac{3nL}{2pv}(\alpha + \cos 2\varphi) + \beta, \quad \text{fietque} \quad dp = \frac{3nL(\alpha + \cos 2\varphi)}{2\,(L+X)}\left(\frac{dp}{pv} + \frac{dv}{vv}\right),$$

ubi dp est quam minimum, et dv involvit excentricitatem q tanquam factorem. Nunc pro expressione dq diminuenda habebimus

$$(L+Y) = 2rdv\left(\frac{3nL(\alpha+\cos 2\varphi)}{2v} + \beta p - pY + vY - \frac{3mL}{2v} - \frac{9nL\cos 2\varphi}{4v}\right) + vdY(pr - 2rv + vv),$$

ubi est $r = \frac{p}{1-qq}$. Statuamus $Y = \zeta + \frac{\eta}{v}$, fietque haec expressio

$$2rdv\left(\frac{3\alpha nL}{2v} - \frac{3nL\cos 2\varphi}{4v} + \beta p - \frac{3mL}{2v} - p\zeta - \frac{p\eta}{v} + v\zeta + \eta\right)$$

$$-\eta dv\left(\frac{pr}{v} - 2r + v\right) + (vd\zeta + d\eta)(pr - 2rv + vv),$$

ut jam termini vdv destruantur, sit $\eta = 2r\zeta$; pro terminis autem dv prodit

$$2r\eta - 2pr\zeta + 2r\eta + 2\beta pr = 0 \quad \text{seu} \quad 2\eta - p\zeta + \beta p = 0,$$

hincque $\beta = \zeta\left(1 - \frac{4r}{p}\right)$. Tum vero termini $\frac{dv}{v}$ tollentur sumendo

$$3\alpha nLr - \tfrac{3}{2}nLr\cos 2\varphi - 3mLr - 3pr\eta = 0, \quad \text{hincque}$$

$$\zeta = \frac{\alpha nL}{2pr} - \frac{nL\cos 2\varphi}{4pr} - \frac{mL}{2pr}.$$

Verum ne variabilitas anguli φ in differentiatione novum momentum introducat, omittamus hic potius terminum $\cos 2\varphi$, ponamusque $\alpha = 0$, ut sit

$$X = \frac{3nL\cos 2\varphi}{2pv} + \frac{mL(4r-p)}{2ppr} \quad \text{et} \quad Y = \frac{-mL(2r+v)}{2prv}.$$

Vel eodem res redibit, si ponamus $\zeta = 0$, $\eta = 0$, $\beta = 0$, ut sit $Y = 0$ et $\alpha n = m$, ideoque

$$X = \frac{3L}{2pv}(m + n\cos 2\varphi), \quad \text{eritque} \quad \frac{v^3 dr}{r}(L+Y) = \frac{-3nLrdv\cos 2\varphi}{2v} \quad \text{seu} \quad \frac{dr}{rr} = \frac{-3ndv\cos 2\varphi}{2v^4}.$$

Deinde vero pro motu lineae absidum habemus in genere

$$d\varphi - ds = \frac{dt\sqrt{2gp}}{vv\sqrt{(L+Y)}}\left(\sqrt{(L+X)(L+Y)} - L - Y\sin^2 s - \frac{pX\cos s}{qv}\right)$$

$$+ \frac{dp\sin s}{qv} - \frac{dp\sin s\cos s}{2p} + \frac{Ydt\cos s\sqrt{2gp}}{q\sqrt{(L+Y)}} + \frac{dY\sin s\cos s}{2(L+Y)}.$$

Cum nunc sit $Y = 0$ et $\sqrt{(L+X)(L+Y)} = L + \frac{1}{2}X$, erit

$$d\varphi - ds = \frac{3dt(m+n\cos 2\varphi)\sqrt{2gLp}}{4pv^3} + \frac{3ndt\cos s\cos 2\varphi\sqrt{2gLp}}{4qv^4} + \frac{dp\sin s}{qv} - \frac{dp\sin s\cos s}{2p}.$$

existente $d\varphi = \frac{dt\sqrt{2gLp}}{vv}\left(1 + \frac{3(m+n\cos 2\varphi)}{4pv}\right)$, unde fit

$$ds = \frac{dt\sqrt{2gLp}}{vv} - \frac{3ndt\cos s\cos 2\varphi\sqrt{2gLp}}{4qv^4} - \frac{dp\sin s}{qv} + \frac{dp\sin s\cos s}{2p}.$$

Est vero $dp = \frac{3L(m+n\cos 2\varphi)}{2(L+X)}\left(\frac{dp}{pv} - \frac{dv}{vv}\right)$ et $dv = \frac{qdt\sin s\sqrt{2gLp}}{p}$, ideoque

$$dp = \frac{3(m + n\cos 2\varphi)\, q dt \sin s \sqrt{2gLp}}{2pvv}.$$

Cum igitur sit proxime $\frac{dt\sqrt{2gLp}}{vv} = ds$, erit $dp = \frac{3(m + n\cos 2\varphi)\, q dt \sin s}{2p}$ atque

$$\frac{dr}{rr} = \frac{-3nqds\sin s\cos 2\varphi}{2pvv} \quad \text{et}$$

$$d\varphi - ds = \frac{3mds}{4pp}\left(1 + 2\sin^2 s + q\cos s + q\sin^2 s\cos s\right) + \frac{3nds\cos 2\varphi}{4pp}\left(1 - 2\cos 2s - \frac{\cos s}{q} + 2q\sin^2 s\cos s \ldots\right.$$

in quibus formulis jam p et q ut constantes et $d\varphi = ds$ spectari possunt. Denique vero ope formulae $d\varphi = \ldots$ omnia ad tempus t revocari poterunt.

Alia solutio ejusdem problematis.

159. Cum ista solutio formulis differentialibus nimium sit implicata, quoniam eae ex differentio-differentialibus sunt immediate deductae, aliam viam tentemus ad hunc casum accommodatam. Cum enim aequationes principales sint

$$\text{I.} \quad v dd\varphi + 2dv d\varphi = \frac{-3ngLdt^2\sin 2\varphi}{v^4},$$

$$\text{II.} \quad ddv - vd\varphi^2 = \frac{-2gLdt^2}{vv} - \frac{3gLmdt^2}{v^4} - \frac{9gLndt^2\cos 2\varphi}{2v^4},$$

prima per v multiplicata, prius membrum integrabile habebit, integrali existente $vvd\varphi$. Integrabilis ergo quoque fiet si multiplicetur per $2v^3 d\varphi$, quo pacto in altero membro elementum dt e signo integrali tolletur; prodibit enim

$$v^4 d\varphi^2 = 2gLdt^2\left(C - 3n\int\frac{d\varphi\sin 2\varphi}{v}\right).$$

Sit brevitatis gratia $\int\frac{d\varphi\sin 2\varphi}{v} = S$, ut habeamus $v^4 d\varphi^2 = 2gLdt^2(C - 3nS)$. Deinde prima per $2vd\varphi$ et altera per $2dv$ multiplicatae, in una summa efficiunt

$$2vvd\varphi dd\varphi + 2vdvd\varphi^2 + 2dvddv = 2gLdt^2\left(-\frac{2dv}{vv} - \frac{3mdv}{v^4} - \frac{3nd\varphi\sin 2\varphi}{v^3} - \frac{9ndv\cos 2\varphi}{2v^4}\right),$$

quae integrata dat

$$dv^2 + vvd\varphi^2 = 2gLdt^2\left(D + \frac{2}{v} + \frac{m}{v^3} + \frac{3n\cos 2\varphi}{2v^3}\right).$$

Cum ergo inde sit $2gLdt^2 = \frac{v^4 d\varphi^2}{C - 3nS}$, hinc commode tempus t eliminatur, obtineturque

$$(C - 3nS)(dv^2 + vvd\varphi^2) = v^4 d\varphi^2\left(D + \frac{2}{v} + \frac{2m + 3n\cos 2\varphi}{2v^3}\right)$$

$$\text{et} \quad d\varphi = \frac{dv\sqrt{(C - 3nS)}}{vv\sqrt{\left(D + \frac{2}{v} + \frac{(2m + 3n\cos 2\varphi)}{2v^3} - \frac{(C - 3nS)}{vv}\right)}}.$$

Jam quoties dv evanescit, necesse est, ut formula irrationalis denominatoris evanescat, quod duobus casibus evenire debeat, quibus angulus quidem s fit vel 0 vel 180^0, denominator factorem habebit $\sin s$. Statuamus ergo $v=\frac{p}{1+q\cos s}$, et denominator erit

$$D+\frac{2}{p}+\frac{2m+3n\cos 2\varphi}{2p^3}-\frac{(C-3nS)}{pp}$$

$$+\frac{2q\cos s}{p}+\frac{3q\cos s\,(2m+3n\cos 2\varphi)}{2p^3}-\frac{2q\cos s\,(C-3nS)}{pp}$$

$$+\frac{3qq\cos^2 s\,(2m+3n\cos 2\varphi)}{2p^3}-\frac{qq\cos^2 s\,(C-3nS)}{pp}$$

$$+\frac{q^3\cos^3 s\,(2m+3n\cos 2\varphi)}{2p^3}.$$

Fiat nunc $$D+\frac{2}{p}+\frac{2m+3n\cos 2\varphi}{2p^3}-\frac{(C-3nS)}{pp}+\frac{3qq(2m+3n\cos 2\varphi)}{2p^3}-\frac{qq\,(C-3nS)}{pp}=0,$$

$$1+\frac{3\,(2m+3n\cos 2\varphi)}{2pp}-\frac{2\,(C-3nS)}{p}+\frac{qq\,(2m+3n\cos 2\varphi)}{2pp}=0,$$

eritque formula irrationalis in denominatore

$$\frac{q\sin s}{p}\sqrt{\left(C-3nS-\frac{3\,(2m+3n\cos 2\varphi)}{2p}-\frac{q\cos s\,(2m+3n\cos 2\varphi)}{2p}\right)}$$

et $$\frac{dv}{vv}=\frac{qd\varphi\sin s}{p}\sqrt{\left(1-\frac{(3+q\cos s)\,(2m+3n\cos 2\varphi)}{2p\,(C-3nS)}\right)}.$$

Jam ex illis aequationibus quantitates p et q definiantur, quae si esset $m=0$ et $n=0$, prodirent: $p=C$ et $qq=1+CD$, atque hi erunt quasi valores medii ipsarum p et q, qui statuantur f et k, ut sit $C=f$ et $D=\frac{kk-1}{f}$. Deinde cum m et n sint quantitates valde parvae, in terminis per m et n affectis scribere licebit $p=f$ et $q=k$, sicque habebimus

$$\frac{1}{p}=\frac{1}{f}+\frac{(3+kk)\,(2m+3n\cos 2\varphi)}{2f^3}+\frac{3nS}{ff}\quad\text{et}\quad\frac{qq}{pp}=\frac{kk}{ff}+\frac{(1+3kk)\,(2m+3n\cos 2\varphi)}{2f^4}+\frac{3nS\,(1+kk)}{f^3},$$

unde fit

$$p=f-\frac{(3+kk)\,(2m+3n\cos 2\varphi)}{4f}-3nS\quad\text{et}\quad qq=kk+\frac{(1-k^4)\,(2m+3n\cos 2\varphi)}{2ff}+\frac{3n\,(1-kk)\,S}{f}.$$

Quoniam nunc habemus valores litterarum p et q, ob $v=\frac{p}{1+q\cos s}$ erit

$$S=\int\frac{d\varphi\,(1+q\cos s)\sin 2\varphi}{p}\quad\text{et}\quad\frac{dv}{vv}=\frac{dp}{pp}-\frac{dq\cos s}{p}+\frac{qds\sin s}{p}+\frac{qdp\cos s}{pp},\quad\text{seu}$$

$$\frac{dv}{vv}=\frac{dp}{pp}-\frac{(pdq-qdp)\cos s}{pp}+\frac{qds\sin s}{p}.$$

At est superioribus formulis differentiandis

$$\frac{dp}{pp}=\frac{3n\,(3+kk)\,d\varphi\sin 2\varphi}{2f^3}-\frac{3n\,(1+q\cos s)\,d\varphi\sin 2\varphi}{ffp},\quad\text{seu}\quad=\frac{3n\,(1-2k\cos s+kk)\,d\varphi\sin\varphi}{2f^3}\quad\text{et}$$

$$\frac{2q\,(pdq-qdp)}{p^3}=\frac{-3n(1+3kk)\,d\varphi\sin 2\varphi}{f^4}+\frac{3n\,(1+kk)\,d\varphi\,(1+k\cos s)\sin 2\varphi}{f^4},$$

$$\frac{pdq - qdp}{pp} = \frac{3nd\varphi \sin 2\varphi\,(1 + kk\cos s - 2k)}{2f^3},$$

ex quibus colligitur

$$\frac{dv}{vv} = \frac{3n(1+kk)\,d\varphi \sin^2 s \sin 2\varphi}{2f^3} + \frac{qds\sin s}{p}.$$

At est

$$\frac{dv}{vv} = \frac{qd\varphi\sin s}{p}\sqrt{\left(1 - \frac{(3+k\cos s)(2m+3n\cos 2\varphi)}{2ff}\right)} = \frac{qd\varphi \sin s}{p} - \frac{k(3+k\cos s)(2m+3n\cos 2\varphi)\,d\varphi\sin s}{4f^3}$$

Ergo $\quad d\varphi - ds = \frac{pd\varphi}{4f^3q}(6n(1+kk)\sin s\sin 2\varphi + k(3+k\cos s)(2m+3n\cos 2\varphi))$.

Hic jam spectatis p et q ut constantibus, nempe $p = f$ et $q = k$, erit

$$d\varphi - ds = \frac{d\varphi}{4ffk}(6mk + 2mkk\cos s + 9nk\cos 2\varphi + 3n(1+\tfrac{3}{2}kk)\cos(2\varphi - s) - 3n(1+\tfrac{1}{2}kk)\cos(2\varphi+s))$$

Cum igitur proxime sit $d\varphi = ds$, erit integrando

$$\varphi - s = \text{Const.} + \frac{3m\varphi}{2ff} + \frac{mk\sin s}{2ff} + \frac{9n\sin 2\varphi}{8ff} + \frac{3n(2+3kk)\sin(2\varphi-s)}{8ffk} - \frac{n(2+kk)\sin(2\varphi+s)}{8ffk},$$

qua aequatione relatio inter longitudinem φ et anomaliam veram s exprimitur. Tum vero quia S per quantitatem minimam n multiplicatur, sufficiet posuisse

$$dS = \frac{d\varphi}{f}(\sin 2\varphi + \tfrac{1}{2}k\sin(2\varphi - s) + \tfrac{1}{2}k\sin(2\varphi+s)),$$

unde fit $\quad S = \frac{-\cos 2\varphi}{2f} - \frac{k\cos(2\varphi-s)}{2f} - \frac{k\cos(2\varphi+s)}{6f},$

hincque deducimus

$$p = f - \frac{m(3+kk)}{2f} - \frac{3n(1+kk)\cos 2\varphi}{4f} + \frac{3nk\cos(2\varphi-s)}{2f} + \frac{3nk\cos(2\varphi+s)}{6f},$$

$$qq = kk + \frac{m(1-k^4)}{ff} + \frac{3nkk(1-kk)\cos 2\varphi}{2ff} - \frac{3nk(1-kk)\cos(2\varphi-s)}{2ff} - \frac{3nk(1-kk)\cos(2\varphi+s)}{6ff}.$$

Denique ut omnia ad tempus t reducamus, habemus

$$dt\sqrt{2gL} = \frac{pp\,d\varphi}{(1+q\cos s)^2\sqrt{(f-3nS)}} = \frac{pp\,d\varphi}{(1+q\cos s)^2}\left(\frac{1}{\sqrt f} + \frac{3nS}{2f\sqrt f}\right),$$

cujus integratio per praecedentes formulas in potestate est censenda. Cum enim sit

$$d\varphi = ds\left(1 + \frac{3m}{2ff} + \frac{mk\cos s}{2ff} + \frac{9n\cos 2\varphi}{4ff} + \frac{3n(2+3kk)\cos(2\varphi-s)}{8ffk} - \frac{3n(2+kk)\cos(2\varphi+s)}{8ffk}\right),$$

$$pp = ff - m(3+kk) - \tfrac{3}{2}n(1+kk)\cos 2\varphi + 3nk\cos(2\varphi-s) + nk\cos(2\varphi+s),$$

$$1 + \frac{3nS}{2f} = 1 - \frac{3n\cos 2\varphi}{4ff} - \frac{3nk\cos(2\varphi-s)}{4ff} - \frac{nk\cos(2\varphi+s)}{4ff}, \quad \text{erit}$$

$$pp\,d\varphi\left(1+\frac{3nS}{2f}\right) = ffds\left(1 - \frac{m(3+2kk)}{2ff} + \frac{mk\cos s}{2ff} - \frac{3nkk\cos 2\varphi}{2ff} + \frac{3n(2+9kk)\cos(2\varphi-s)}{8ffk} - \frac{3n(2-kk)\cos(2\varphi+s)}{8ffk}\right)$$

Porro est

$$q = k + \frac{m(1-k^4)}{2ffk} + \frac{3nk(1-kk)\cos 2\varphi}{4ff} - \frac{3n(1-kk)\cos(2\varphi-s)}{4ff} - \frac{n(1-kk)\cos(2\varphi+s)}{4ff},$$

tandem concluditur $dt\sqrt{2fgL} = \frac{ffds}{(1+k\cos s)^2} + \frac{ffWds}{(1+k\cos s)^3}$ existente

$$W = \frac{-3m(2+kk)}{4ff} - \frac{m(1+kk)\cos s}{ffk} + \frac{mkk\cos 2s}{4ff} + \frac{n(8-5kk)\cos 2\varphi}{8ff}$$

$$+ \frac{3n(2+7kk)\cos(2\varphi - s)}{8ffk} + \frac{3n(6+5kk)\cos(2\varphi-2s)}{16ff}$$

$$- \frac{3n(2+kk)\cos(2\varphi+s)}{8ffk} - \frac{n(2+kk)\cos(2\varphi+2s)}{16ff}.$$

160. **Coroll. 1.** Formula $\varphi - s$ exprimit longitudinem absidis imae, unde si corpus N nunc fuerit in abside ima, ad absidem summam pertinget confecto angulo φ, ut ob $s = 180^0 = \pi$ sit

$$\varphi - \pi = \frac{3m\pi}{2ff} + \frac{9n\sin 2\varphi}{8ff} + \frac{3n(2+3kk)\sin(2\varphi-\pi)}{8ffk} - \frac{n(2+kk)\sin(2\varphi+\pi)}{8ffk},$$

$$\text{seu}\quad \varphi - \pi = \frac{3m\pi}{2ff} + \frac{9n\sin 2\varphi}{8ff} - \frac{n(1+2kk)\sin 2\varphi}{2ffk},$$

ubi ob $2\varphi = 2\pi$ proxime, et neglectis terminis binas dimensiones litterarum m et n involventibus, erit $\varphi = \pi + \frac{3m\pi}{2ff}$.

161. **Coroll. 2.** At dum absolvitur anomalia vera $s = 2\lambda\pi$, existente λ numero integro valde magno, ob $\varphi = 2\lambda\pi + \frac{3\lambda m\pi}{ff}$, proxime erit

$$\varphi = 2\lambda\pi + \frac{3\lambda m\pi}{ff} + \frac{9n}{8ff}\sin\frac{6\lambda m\pi}{ff} + \frac{n(1+2kk)}{2ffk}\sin\frac{6\lambda m\pi}{ff};$$

unde si sit $\frac{6\lambda m}{ff} = \frac{1}{2}$, seu $\lambda = \frac{ff}{12m}$, post $\frac{ff}{2m}$ revolutiones anomaliae verae, erit

$$\varphi = 2\lambda\pi + \tfrac{1}{4}\pi + \frac{9n}{4ff} + \frac{n(1+2kk)}{2ffk}.$$

162. **Coroll. 3.** Si esset $n = 0$, promotio lineae absidum in singulis revolutionibus anomaliae verae s foret eadem, scilicet $= \frac{3m\pi}{ff} = \frac{3(cc-aa)}{ff}\pi$, uti jam supra invenimus. Sed si n non est $= 0$, singulis revolutionibus anomaliae verae non amplius aequalis progressio lineae absidum respondet, quod tamen discrimen demum post plures revolutiones fit sensibile.

163. **Coroll. 4.** Relatio inter angulos φ et s ita definitur, ut sit

$$\varphi = \zeta + s + \frac{3ms}{2ff} + \frac{mk\sin s}{2ff} + \frac{9n\sin 2\varphi}{8ff} + \frac{3n(2+3kk)\sin(2\varphi - s)}{8ffk} - \frac{n(2+kk)\sin(2\varphi+s)}{8ffk},$$

ubi in posterioribus terminis pro φ scribi potest $\zeta + s + \frac{3ms}{2ff}$, neque vero hic terminum $\frac{3ms}{2ff}$ omittere licet, cum is crescente cum tempore angulo s ad valorem quantumvis magnum assurgere possit. Constans autem ζ non est arbitraria, sed denotat longitudinem absidis imae ab axe principali JA.

164. Scholion 1. Pro quavis ergo anomalia vera s et angulo constante ζ defini[...] longitudo corporis N seu angulus $AJN = \varphi$, qua cognita porro semiparameter p et excent[...] orbitae ellipticae variabilis innotescit, unde concluditur distantia $JN = \varrho = \frac{p}{1 + q \cos s}$. Supp[...] autem, ut relatio inter tempus t et angulum s assignetur, seu ut haec aequatio

$$dt \sqrt{2fgL} = \frac{ffds}{(1 + k \cos s)^2} + \frac{ffWds}{(1 + k \cos s)^3}$$

integretur, quod negotium, quia φ per s datur, concedendum est. Est enim

$$\int \frac{ds}{1 + k \cos s} = \frac{1}{\sqrt{(1 - kk)}} \text{Arc.} \cos \frac{k + \cos s}{1 + k \cos s},$$

$$\int \frac{ds}{(1 + k \cos s)^2} = \frac{1}{(1 - kk)^{\frac{3}{2}}} \text{Arc.} \cos \frac{k + \cos s}{1 + k \cos s} - \frac{k \sin s}{(1 - kk)(1 + k \cos s)},$$

$$\int \frac{ds}{(1 + k \cos s)^3} = \frac{2 + kk}{2(1 - kk)^{\frac{5}{2}}} \text{Arc.} \cos \frac{k + \cos s}{1 + k \cos s} - \frac{k \sin s}{2(1 - kk)(1 + k \cos s)^2} - \frac{3k \sin s}{2(1 - kk)^2 (1 + k \cos s)}.$$

Reliquae partes exigunt integrationem hujusmodi formulae $\int \frac{ds \cos (\alpha s + \beta)}{(1 + k \cos s)^3}$, quae si k fuerit fractio valde parva, facile per seriem evolvitur; posito enim

$$\frac{1}{(1 + k \cos s)^3} = A + B \cos s + C \cos 2s + D \cos 3s + E \cos 4s + \text{etc.}$$

reperitur integrale $\int \frac{ds \cos (\alpha s + \beta)}{(1 + k \cos s)^3}$ ita expressum

$$\frac{A}{\alpha} \sin (\alpha s + \beta) + \frac{B \sin (\alpha s - s + \beta)}{2(\alpha - 1)} + \frac{C \sin (\alpha s - 2s + \beta)}{2(\alpha - 2)} + \text{etc.}$$
$$+ \frac{B \sin (\alpha s + s + \beta)}{2(\alpha + 1)} + \frac{C \sin (\alpha s + 2s + \beta)}{2(\alpha + 2)} + \text{etc.}$$

verum nisi k sit fractio valde parva, haec series parum juvat.

165. Scholion 2. At si k est quantitas valde exigua, aliud incommodum nascitur, quod in nostris formulis termini per k divisi nimis fiant magni, ideoque determinatio anguli $\varphi - s$ nimis incerta reddatur. Operae ergo pretium erit casum, quo $k = 0$, data opera evolvisse, quo fit

$$p = f - \frac{3(2m + 3n \cos 2\varphi)}{4f} - 3nS \quad \text{et} \quad qq = \frac{2m + 3n \cos 2\varphi}{2ff} + \frac{3nS}{f},$$

quae formula autem locum habere nequit, nisi sit positiva; si enim fieret negativa, hoc indicio esset quantitatem k evanescere non posse, vel tum etiam $\cos s$ imaginarium esse proditurum, ita ut hoc casu positio $\varrho = \frac{p}{1 + k \cos s}$ contradictionem involveret. Casu autem, quo qq prodit positivum, reperitur

$$d\varphi = ds + \frac{3nd\varphi \sin s \sin 2\varphi}{f\sqrt{(4m + 6n \cos 2\varphi + 12fnS)}} \text{ existente } S = \int \frac{d\varphi \sin 2\varphi}{f} = \frac{-\cos 2\varphi}{2f},$$

ita ut sit

Erit $n d\varphi = ds + \frac{3 n d\varphi \sin s \sin 2\varphi}{2f\sqrt{m}}$ et $qq = \frac{m}{ff}$ et $p = f - \frac{3m}{2f} - \frac{3n\cos 2\varphi}{4f}$

Itaque $q = \frac{\sqrt{m}}{f}$, sicque excentricitas q constans. Tum erit

$$dt\sqrt{2fgL} = \frac{d\varphi\,(ff - 3m - \frac{9}{4}n\cos 2\varphi)}{\left(1 + \frac{\sqrt{m}}{f}\cos s\right)^2}, \text{ seu } = d\varphi\,(ff - 2f\cos s\sqrt{m} - 3m - \tfrac{9}{4}n\cos 2\varphi).$$

Solutio ergo hujus casus pendet a resolutione hujus aequationis $d\varphi = ds + \frac{3 n d\varphi \sin s \sin 2\varphi}{2f\sqrt{m}}$, ex qua, si $\frac{n}{f\sqrt{m}}$ est quantitas valde parva, concluditur

$$\varphi = \zeta + s + \frac{3n\sin(2\varphi - s)}{4f\sqrt{m}} - \frac{n\sin(2\varphi + s)}{4f\sqrt{m}}.$$

Reliquis autem casibus, praecipue si m esset $= 0$, alia tractatio requireretur, in valorem scilicet ipsius S accuratius inquiri oporteret, quod difficultatibus haud esset cariturum.

166. **Scholion 3.** Solutio nostri problematis posterior ideo priori est anteferenda, quod binarum aequationum differentio-differentialium propositarum una integratio successerit. In genere igitur, si idem usu veniat, solutio facilior obtineri potest. Propositis enim his duabus aequationibus

$$\varrho dd\varphi + 2d\varrho d\varphi = -gLTdt^2 \quad \text{et} \quad dd\varrho - \varrho d\varphi^2 = -gLdt^2\left(\frac{2}{\varrho\varrho} + V\right),$$

multiplicetur prior per $2\varrho^3 d\varphi$, ut prodeat

$$\varrho^4 d\varphi^2 = 2gLdt^2\,(C - \int T\varrho^3 d\varphi) = 2gLdt^2\,(C - S),$$

posito $\int T\varrho^3 d\varphi = S$. Deinde priori per $2\varrho d\varphi$, et posteriori per $2d\varrho$ multiplicata, summa praebet

$$d\,.\,(\varrho\varrho d\varphi^2 + d\varrho^2) = -2gLdt^2\left(T\varrho d\varphi + Vd\varrho + \frac{2d\varrho}{\varrho\varrho}\right).$$

Quod si jam fuerit $T\varrho d\varphi + Vd\varrho$ integrabile, ponatur integrale $\int(T\varrho d\varphi + Vd\varrho) = \frac{R}{\varrho^3}$, ut habeamus

$$d\varrho^2 + \varrho\varrho d\varphi^2 = 2gLdt^2\left(D + \frac{2}{\varrho} - \frac{R}{\varrho^3}\right);$$

hinc eliminando dt^2 adipiscemur

$$(C - S)\,d\varrho^2 = \varrho^4 d\varphi^2\left(D + \frac{2}{\varrho} - \frac{R}{\varrho^3} - \frac{(C-S)}{\varrho\varrho}\right) \quad \text{et} \quad \frac{d\varrho}{\varrho\varrho}\sqrt{(C - S)} = d\varphi\sqrt{\left(D + \frac{2}{\varrho} - \frac{(C-S)}{\varrho\varrho} - \frac{R}{\varrho^3}\right)}.$$

Statuamus $\varrho = \frac{p}{1 + q\cos s}$, sitque

$$D + \frac{2}{p} - \frac{R(1 + 3qq)}{p^3} - \frac{(1 + qq)(C - S)}{pp} = 0 \quad \text{et} \quad 2 - \frac{R(3 + qq)}{pp} - \frac{2(C - S)}{p} = 0,$$

ut fiat formula irrationalis

$$\sqrt{\left(D + \frac{2}{\varrho} - \frac{(C-S)}{\varrho\varrho} - \frac{R}{\varrho^3}\right)} = \frac{q\sin s}{p}\sqrt{\left(C - S + \frac{R(3 + q\cos s)}{p}\right)}, \text{ hincque}$$

$$\frac{d\varrho}{\varrho\varrho} = \frac{q d\varphi \sin s}{p}\sqrt{\left(1 + \frac{R(3 + q\cos s)}{p\,(C - S)}\right)}.$$

*

Inde autem cum R et S sint quantitates valde parvae, posito $C=f$ et $D=\frac{kk-1}{f}$, ut fiat proxime $p=f$ et $q=k$, colligitur

$$\frac{1}{p}=\frac{1}{f}+\frac{S}{ff}-\frac{(3+kk)R}{2f^2} \quad \text{et} \quad p=f-S+\frac{(3+kk)R}{2f},$$

$$\frac{qq}{pp}=\frac{kk}{ff}+\frac{(1+kk)S}{f^3}-\frac{(1+3kk)R}{f^4},$$

unde fit
$$qq=kk+\frac{(1-kk)S}{f}-\frac{(1-k^4)R}{ff}.$$

Deinde ob $\quad \frac{dp}{pp}=\frac{-dS}{ff}+\frac{(3+kk)dR}{2f^3} \quad$ et $\quad \frac{pdq-qdp}{pp}=\frac{(1+kk)dS}{2ffk}-\frac{(1+3kk)dR}{2f^3k}, \quad$ erit

$$\frac{dv}{vv}=\frac{qdS\sin s}{p}-\frac{dS}{ff}-\frac{(1+kk)dS\cos S}{2ffk}+\frac{(3+kk)dR}{2f^3}+\frac{(1+3kk)dR\cos s}{2f^3k}.$$

Est vero etiam $\frac{dv}{vv}=\frac{qd\varphi\sin s}{p}\left(1+\frac{(3+k\cos s)R}{2ff}\right)$, unde

$$\frac{q\sin s}{p}(d\varphi-ds)=\frac{-dS}{ff}-\frac{(1+kk)dS\cos s}{2ffk}+\frac{(3+kk)dR}{2f^3}+\frac{(1+3kk)dR\cos s}{2f^3k}-\frac{k(3+k\cos s)R\sin s}{2f^3}d\varphi.$$

Denique est $\quad dt\sqrt{2fgL}=vvd\varphi\left(1+\frac{S}{f}\right)=\frac{ppd\varphi}{(1+q\cos s)^2}\left(1+\frac{S}{f}\right);$

ubi notandum est esse ob $dv=\frac{kvvd\varphi\sin s}{f}$ in terminis minimis

$$dS=Tv^3d\varphi \quad \text{et} \quad dR=\frac{3kRvd\varphi\sin s}{f}+Tv^4d\varphi+\frac{kVv^5d\varphi\sin s}{f},$$

unde fit

$$\frac{q}{p}(d\varphi-ds)=\frac{(1+kk)Tv^4\sin s}{2f^3}d\varphi+\frac{Rvd\varphi}{2f^4}(6k+(3+5kk)\cos s+3k^3-k^3\cos^2 s)$$
$$+\frac{Vv^4d\varphi}{2f^4}(k(3+kk)+(1+3kk)\cos s).$$

167. **Scholion 4.** Aliam formam habitura esset solutio, si formula integralis

$$\int(Tvd\varphi+Vdv) \quad \text{non} \quad \frac{R}{v^3}, \quad \text{sed} \quad \frac{R}{v^\lambda},$$

vel aggregato ex pluribus hujusmodi formulis aequalis poneretur. Ponamus ergo

$$\int Tvd\varphi+Vdv)=-\mathfrak{A}-\frac{\mathfrak{B}}{v}-\frac{\mathfrak{C}}{vv}-\frac{D}{v^3}-\frac{E}{v^4}-\frac{F}{v^5}-\text{etc.}$$

existente $\quad \int(Tv^3d\varphi=S \quad \text{et} \quad dt\sqrt{2fgL}=vvd\varphi\left(1+\frac{S}{2f}\right),$

habebimus ergo

$$\frac{dv}{vv}\sqrt{(f-S)}=d\varphi\sqrt{\left(\frac{kk-1}{f}+\mathfrak{A}+\frac{2-\mathfrak{B}}{v}-\frac{(f-S-\mathfrak{C})}{vv}+\frac{D}{v^3}+\frac{E}{v^4}+\frac{F}{v^5}+\text{etc.}\right)},$$

quantitates $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, D, E, F et S ut valde parvae sunt spectandae. Ponamus brevitatis gratia

$$\frac{kk-1}{f}+\mathfrak{A}=A,\quad 2+\mathfrak{B}=B,\quad \text{et}\quad -f+S+\mathfrak{C}=C,$$

ut formula irrationalis sit

$$\sqrt{\left(A+\frac{B}{v}+\frac{C}{vv}+\frac{D}{v^3}+\frac{E}{v^4}+\frac{F}{v^5}\right)},$$

quae posito $v=\frac{p}{1+q\cos s}$ ita comparata esse debet, ut factorem obtineat $\sin s$, seu ut evanescat posito tam $s=0$ quam $s=180^0$. Quocirca efficiendum est, ut fiat

$$A+B\left(\frac{1\pm q}{p}\right)+C\left(\frac{1\pm q}{p}\right)^2+D\left(\frac{1\pm q}{p}\right)^3+\text{etc.}=0,$$

ergò necesse est $\frac{1+q}{p}$ et $\frac{1-q}{p}$ binae radices hujus aequationis

$$A+Bz+Czz+Dz^3+Ez^4+Fz^5+\text{etc.}=0,$$

quae rejectis terminis minimis habebit hanc formam $\frac{kk-1}{f}+2z-fzz=0$, unde fit $z=\frac{1\pm k}{f}$, ita ut sit proxime $p=f$ et $q=k$. Ponatur jam in terminis minimis $p=f$ et $q=k$, et habebimus

$$\frac{kk-1}{f}+\mathfrak{A}+\frac{2(1\pm q)}{p}+\frac{\mathfrak{B}(1\pm k)}{f}-\frac{f(1\pm q)^2}{pp}+\frac{(S+\mathfrak{C})(1\pm k)^2}{ff}+\frac{D(1\pm k)^3}{f^3}+\frac{E(1\pm k)^4}{f^4}+\frac{F(1\pm k)^5}{f^5}=0,$$

quae ob signa ambigua resolvitur in has duas

$$\frac{kk-1}{f}+\mathfrak{A}+\frac{2}{p}+\frac{\mathfrak{B}}{f}-\frac{f(1+qq)}{pp}+\frac{(S+\mathfrak{C})(1+kk)}{ff}+\frac{D(1+3kk)}{f^3}+\frac{E(1+6kk+k^4)}{f^4}+\frac{F(1+10kk+5k^4)}{f^5}=0,$$

$$\frac{2q}{p}+\frac{\mathfrak{B}k}{f}-\frac{2fq}{pp}+\frac{2(S+\mathfrak{C})k}{ff}+\frac{D(3k+k^3)}{f^3}+\frac{E(4k+4k^3)}{f^4}+\frac{F(5k+10k^3+k^5)}{f^5}=0.$$

Ponamus jam $\frac{1}{p}=\frac{1+x}{f}$, et prior aequatio abit in hanc

$$\frac{kk}{f}+\mathfrak{A}+\frac{\mathfrak{B}}{f}-\frac{fqq}{pp}+\frac{(S+\mathfrak{C})(1+kk)}{ff}+\frac{D(1+3kk)}{f^3}+\text{etc.}=0,$$

unde deducimus

$$\frac{qq}{pp}=\frac{kk}{ff}+\frac{\mathfrak{A}}{f}+\frac{\mathfrak{B}}{ff}+\frac{(S+\mathfrak{C})(1+kk)}{f^3}+\frac{D(1+3kk)}{f^4}+\frac{E(1+6kk+k^4)}{f^5}+\frac{F(1+10kk+5k^4)}{f^6}+\text{etc.};$$

altera autem per q multiplicata, qui factor in terminis minimis abit in k, praebet

$$\frac{2x}{f}=\frac{\mathfrak{B}}{f}+\frac{2(\mathfrak{C}+S)}{ff}+\frac{D(3+kk)}{f^3}+\frac{E(4+4kk)}{f^4}+\frac{F(5+10kk+k^4)}{f^5},$$

unde deducimus

$$\frac{1}{p}=\frac{1}{f}+\frac{\mathfrak{B}}{2f}+\frac{\mathfrak{C}+S}{ff}+\frac{D(3+kk)}{2f^3}+\frac{E(4+4kk)}{2f^4}+\frac{F(5+10kk+k^4)}{2f^5}+\text{etc.}$$

$$p=f\left(1-\frac{\mathfrak{B}}{2}-\frac{\mathfrak{C}-S}{f}-\frac{D(3+kk)}{2ff}-\frac{E(4+4kk)}{2f^3}-\frac{F(5+10kk+k^4)}{2f^4}-\text{etc.}\right)$$

$$qq = kk + \mathfrak{A}f + \mathfrak{B}(1 - kk) + \frac{(\mathfrak{C} + S)(1 - kk)}{f} + \frac{D(1 - k^4)}{ff} + \frac{E(1 + 2kk - 3k^4)}{f^3} + \frac{F(1 + 5kk \ldots}{f^4}$$

Tum autem formula irrationalis induit hanc formam

$$\frac{q \sin s}{p} \sqrt{\Big(f - S - \mathfrak{C} - \frac{D(3 + k\cos s)}{f} - \frac{E(6 + 4k\cos s + kk(1 + \cos^2 s))}{ff}}$$

$$- \frac{F(10 + 10k\cos s + 5kk(1 + \cos^2 s) + k^3 \cos s (1 + \cos^2 s))}{f^3}\Big),$$

unde concludimus

$$\frac{dv}{vv} = \frac{q\,d\varphi \sin s}{p} - \frac{k\,d\varphi \sin s}{2ff}\Big(\mathfrak{C} + \frac{D(3 + k\cos s)}{f} + \frac{E(6 + 4k\cos s + kk(1 + \cos^2 s))}{ff} + \text{etc.}\Big).$$

Est vero etiam $\frac{dv}{vv} = \frac{qds \sin s}{p} + \frac{dp}{pp} - \frac{(pdq - qdp)}{pp}\cos s$, unde

$$\frac{q \sin s}{p}(d\varphi - ds) = \frac{k\,d\varphi \sin s}{2ff}\Big(\mathfrak{C} + \frac{D(3 + k\cos s)}{f} + \frac{E(6 + 4k\cos s + kk(1 + \cos^2 s))}{ff} + \frac{F(10 + 10k\cos s + kk(5 + k\cos s)(1 + \ldots}{f^3}$$

$$+ \frac{dp}{pp} - \frac{(pdq - qdp)\cos s}{pp}$$

ubi quidem haec differentialia ipsarum dp et dq non tam commode exprimere licet, quam ... Quoties autem unico termino constat integrale $\int(Tv\,d\varphi + V\,dv)$, toties posterius membrum ... potest ad formam per $k \sin s$ multiplicatam. Est autem in genere

$$\frac{dp}{pp} - \frac{(pdq - qdp)\cos s}{pp} = \frac{-Tv^3 d\varphi}{2ffk}(2k + (1 + kk)\cos s) - \frac{d\mathfrak{A}\cos s}{2k} - \frac{d\mathfrak{B}(k + \cos s)}{2fk} - \frac{d\mathfrak{C}(2k + (1 + kk)\cos s)}{2ffk}$$

$$- \frac{dD(3k + k^3 + (1 + 3kk)\cos s)}{2f^3 k} - \frac{dE(4k + 4k^3 + (1 + 6kk + k^4)\cos s)}{2f^4 k} - \frac{dF(5k + 10k^3 + k^5 + (1 + 10kk + 5k^4)\cos s)}{2f^5 k} \text{ etc.}$$

quae expressio transmutatur in hanc formam

$$\frac{-(2k + (1 + kk)\cos s)\,vv}{2ffk}\Big(Tv\,d\varphi + d\mathfrak{A} + \frac{d\mathfrak{B}}{v} + \frac{d\mathfrak{C}}{vv} + \frac{dD}{v^3} + \frac{dE}{v^4} + \frac{dF}{v^5} \text{ etc.}\Big)$$

$$+ \frac{vv \sin^2 s}{2ff}\left\{\begin{array}{l} d\mathfrak{A}\left(1 + \dfrac{f}{v}\right) + \dfrac{d\mathfrak{B}}{v} - \dfrac{dD}{fvv}(1 + kk) - \dfrac{dE}{ffvv}\left(1 + 3kk + \dfrac{(1 + kk)f}{v}\right) \\ - \dfrac{dF}{f^3 vv}\left(1 + 6kk + k^4 + \dfrac{(1 + 3kk)f}{v} + \dfrac{(1 + kk)ff}{vv}\right) \end{array}\right\}$$

At ex aequatione assumta est

$$d\mathfrak{A} + \frac{d\mathfrak{B}}{v} + \frac{d\mathfrak{C}}{vv} + \frac{dD}{v^3} + \text{etc.} = -Tv\,d\varphi - V\,dv + \frac{\mathfrak{B}\,dv}{vv} + \frac{2\mathfrak{C}\,dv}{v^3} \text{ etc.}$$

ita ut prius membrum superioris aequationis abeat in

$$+ \frac{(2k + (1 + kk)\cos s)}{2ffk} vv\,dv\Big(V - \frac{\mathfrak{B}}{vv} - \frac{2\mathfrak{C}}{v^3} - \frac{3D}{v^4} - \frac{4E}{v^5} - \frac{5F}{v^6} - \text{etc.}\Big).$$

Quare cum in terminis his minimis sit $dv = \frac{kd\varphi \sin s}{f} vv$, evidens est totam aequationem praecedentem dividi posse per sin s; reperitur enim

$$d\varphi - ds = \frac{d\varphi}{2f}\Big(\mathfrak{C} + \frac{D(3+k\cos s)}{f} + \frac{E(6+kk+4k\cos s+kk\cos^2 s)}{ff} + \frac{F(10+5kk+k(10+kk)\cos s+5kk\cos^2 s+k^3\cos^3 s)}{f^3}\Big)$$

$$+\frac{(2k+(1+kk)\cos s)}{2ffk} v^4 d\varphi \Big(V - \frac{\mathfrak{B}}{vv} - \frac{2\mathfrak{C}}{v^3} - \frac{3D}{v^4} - \frac{4E}{v^5} - \frac{5F}{v^6} - \text{etc.}\Big) +$$

$$\Big(d\mathfrak{A} + \frac{f}{v}(d\mathfrak{A} + \frac{d\mathfrak{B}}{f}) - \frac{(1+kk)}{fvv}(dD + \frac{dE}{v} + \frac{dF}{vv} + \text{etc.}) - \frac{(1+3kk)}{ffvv}(dE + \frac{dF}{v} + \text{etc.}) - \frac{(1+6kk+k^4)}{f^3 vv}(dF + \text{etc.})\Big)$$

Atque haec est methodus generalis hujusmodi problemata tractandi, quoties formula $Tv\,d\varphi + V dv$ est integrabilis. Deinde etiam pro formula $\int Tv^3 d\varphi$, quam posuimus $= S$, sive sit integrabilis sive minus, poni poterit $\frac{S}{v^\lambda}$, pro numero dimensionum, quas v in ea obtinet, unde solutio saepe commodius reddi potest, ad quod haec solutio pariter extenditur.

Tertia solutio problematis propositi.

§ 168. Instituantur omnia ut in solutione secunda § 159, sed ponatur

$$\int \frac{d\varphi \sin 2\varphi}{v} = \frac{Q}{v} \quad \text{erit} \quad v^4 d\varphi^2 = 2gL dt^2 \left(f - \frac{3nQ}{v}\right),$$

ac porro per integrationem

$$\frac{dv}{vv}\sqrt{\left(f - \frac{3nQ}{v}\right)} = d\varphi \sqrt{\left(\frac{kk-1}{f} + \frac{2}{v} - \frac{f}{vv} + \frac{2m+3n\cos 2\varphi + 6nQ}{2v^3}\right)}.$$

Hinc posito $v = \frac{p}{1+q\cos s}$, ut fiat

$$\frac{dv}{vv}\sqrt{\left(f - \frac{3nQ}{v}\right)} = \frac{qd\varphi \sin s}{p}\sqrt{\left(f - \frac{(3+q\cos s)(2m+3n\cos 2\varphi + 6nQ)}{2p}\right)},$$

seu quia Q est valde parvum,

$$\frac{dv}{vv} = \frac{qd\varphi \sin s}{p}\sqrt{\left(1 - \frac{6nQ}{fp} - \frac{(2m+3n\cos 2\varphi)(3+q\cos s)}{2fp}\right)},$$

statui debet

$$\frac{1}{p} = \frac{1}{f} + \frac{(3+kk)(2m+3n\cos 2\varphi + 6nQ)}{2f^3} \quad \text{et} \quad \frac{qq}{pp} = \frac{kk}{ff} + \frac{(1+3kk)(2m+3n\cos 2\varphi + 6nQ)}{2f^4},$$

unde fit

$$p = f - \frac{(3+kk)(2m+3n\cos 2\varphi + 6nQ)}{4f} \quad \text{et} \quad qq = kk + \frac{(1-k^4)(2m+3n\cos 2\varphi + 6nQ)}{2ff}.$$

Cum nunc sit $vdQ - Qdv = vd\varphi \sin 2\varphi$, ideoque

$$dQ = d\varphi \sin 2\varphi + \frac{Qdv}{v} = d\varphi \sin 2\varphi + \frac{kQv d\varphi \sin s}{f},$$

quia in terminis minimis est $dv = \frac{kvv d\varphi \sin s}{f}$, erit

$$\frac{dp}{pp} = \frac{-3nkQ\nu d\varphi \sin s\,(3+kk)}{f^4}, \quad \frac{2q\,(pdq-qdp)}{p^3} = \frac{-3nkQ\nu d\varphi \sin s\,(1+3kk)}{f^5}$$

ob $\frac{d\nu}{\nu\nu} = \frac{dp}{pp} - \frac{(pdq-qdp)\cos s}{pp} + \frac{qds \sin s}{p}$, habebimus

$$\frac{d\nu}{\nu\nu} = \frac{qds \sin s}{p} - \frac{3nQ\nu d\varphi \sin s}{2f^4}(2k\,(3+kk) - (1+3kk)\cos s).$$

Est vero ex superioribus

$$\frac{d\nu}{\nu\nu} = \frac{qd\varphi \sin s}{p} - \frac{3nkQd\varphi \sin s}{f^3} - \frac{k\,(2m+3n\cos 2\varphi)\,(3+k\cos s)\,d\varphi \sin s}{4f^3},$$

unde concludimus

$$d\varphi - ds = \frac{d\varphi\,(2m+3n\cos 2\varphi)\,(3+k\cos s)}{4ff} - \frac{3nQ\nu d\varphi}{2f^3 k}(2k\,(2+kk) - (1+5kk)\cos s).$$

Cum jam sit $\frac{Q}{\nu} = \frac{1}{f}\int d\varphi\,(\sin 2\varphi + \frac{1}{2}k \sin(2\varphi - s) + \frac{1}{2}k \sin(2\varphi + s))$, et proxime $d\varphi = ds$,

$$\frac{Q}{\nu} = \frac{-\cos 2\varphi}{2f} - \frac{k\cos(2\varphi - s)}{2f} - \frac{k\cos(2\varphi + s)}{6f} \quad \text{et}$$

$$\frac{\cos 2\varphi + 2Q}{\nu} = \frac{-k\cos(2\varphi - s)}{2f} + \frac{k\cos(2\varphi + s)}{6f} = \frac{-k}{3f}(\cos 2\varphi \cos s + 2\sin 2\varphi \sin s),$$

hincque
$$p = f - \frac{m\,(3+kk)}{2f} - \frac{nk\,(3+kk)\,(\cos(2\varphi + s) - 3\cos(2\varphi - s))}{8f\,(1+k\cos s)},$$

$$qq = kk + \frac{m\,(1-k^4)}{ff} + \frac{nk\,(1-k^4)\,(\cos(2\varphi + s) - 3\cos(2\varphi - s))}{4ff\,(1+k\cos s)}.$$

Invento valore ipsius Q, accuratius relatio inter $d\varphi$ et ds definitur, indeque vera relatio inter φ et s, qua cognita habebitur

$$dt\sqrt{2fgL} = \frac{\nu\nu d\varphi}{\sqrt{(1 + \frac{n}{2ff}(3\cos 2\varphi + 3k\cos(2\varphi - s) + k\cos(2\varphi + s))}}, \quad \text{seu}$$

$$dt\sqrt{2fgL} = \frac{ppd\varphi}{(1+q\cos s)^2} - \frac{nd\varphi\,(3\cos 2\varphi + 3k\cos(2\varphi - s) + k\cos(2\varphi + s))}{4ff\,(1+k\cos s)^2}.$$

Verum haec solutio minus idonea videtur quam secunda.

169. **Problema.** (Fig 183.) Si corpus N circa punctum quasi fixum J non in eodem plano moveatur, ad quod, praeter vim quadratis distantiarum reciproce proportionalem, sollicitetur viribus exiguis quibuscunque, ejus motum tam in longitudinem quam in latitudinem definire.

Solutio. Referatur motus ad planum fixum AJB, in quo sumta recta fixa JA, sint coordinatae orthogonales $JX = x$, $XY = y$ et $YZ = z$, ac ponatur distantia $JN = \sqrt{(xx+yy+zz)} = \dots$ Quibus positis sumtoque elemento temporis dt constante, motus hujusmodi tribus aequationibus exprimetur:

$$ddx = -2gLdt^2\left(\frac{x}{\rho^3} + X\right)$$

$$ddy = -2gLdt^2\left(\frac{y}{\rho^3} + Y\right)$$

$$ddz = -2gLdt^2\left(\frac{z}{\rho^3} + Z\right),$$

ubi quantitates X, Y, Z ut valde parvae sunt spectandae. Consideretur elementum Nn seu directio motus, in qua nunc corpus movetur, quae cum puncto fixo J continet planum, cujus intersectio cum plano assumto AJB sit recta $J☊$, quae vocatur linea nodorum, ac terminus quidem ☊ nodus ascendens, ubi corpus supra planum AJB ascendere incipit. Hic duae res notandae occurrunt, primo longitudo nodi ascendentis seu angulus $AJ☊ = \psi$ et inclinatio plani $☊JN$ ad planum fixum AJB, quae sit $= \omega$. Ex Y ad $J☊$ ducatur normalis $Y☊$, junctaque $N☊$, quae etiam ad $J☊$ erit normalis, fiet angulus $Y☊N = \omega$. Statuatur nunc angulus $☊JN = \sigma$, erit $N☊ = \rho \sin\sigma$ et $J☊ = \rho\cos\sigma$, hincque $YN = \rho\sin\sigma\sin\omega = z$ et $☊Y = \rho\sin\sigma\cos\omega$, unde ob $XY☊ = AJ☊ = \psi$, concluditur $x = \rho\cos\sigma\cos\psi - \rho\sin\sigma\cos\omega\sin\psi$ et $y = \rho\cos\sigma\sin\psi + \rho\sin\sigma\cos\omega\cos\psi$. Quo autem facilius relationem inter hos angulos σ, ω, ψ eorumque differentialia investigemus, re ad trigonometriam sphaericam perducta, sit (fig. 183) arcus $A☊ = \psi$, $☊\omega = d\psi$, $☊N = \sigma$, angulus $B☊N = \omega$, $B\omega n = \omega + d\omega$, et $\omega n = \sigma + d\sigma$. Ducto $\omega\pi$ perpendiculo in $☊N$ erit $☊\pi = d\psi\cos\omega$, et ob $\omega Y = \pi N$ habebimus $\sigma - d\psi\cos\omega = \sigma + d\sigma - Nn$, unde fit $d\sigma = Nn - d\psi\cos\omega$. Tum vero est

$$\sin\omega : \sin(\omega + d\omega) = \sin(\sigma - d\psi\cos\omega) : \sin\sigma, \text{ seu } \sin\omega : \sin\omega + d\omega\cos\omega = \sin\sigma - d\psi\cos\sigma\cos\omega : \sin\sigma,$$

hincque dividendo $\sin\omega : d\omega\cos\omega = \sin\sigma : d\psi\cos\sigma\cos\omega$, unde fit

$$d\omega\sin\sigma = d\psi\cos\sigma\sin\omega \quad \text{seu} \quad d\omega = \frac{d\psi\cos\sigma\sin\omega}{\sin\sigma}.$$

His notatis resumamus nostras aequationes differentio-differentiales ex quibus concludimus (fig. 183)

$$dx^2 + dy^2 + dz^2 = 2gLdt^2\left(D + \frac{2}{\rho} - 2\int(Xdx + Ydy + Zdz)\right),$$

ubi est $Nn = \sqrt{(dx^2 + dy^2 + dz^2)}$. At est angulus elementaris

$$NJn = \frac{\sqrt{(dx^2 + dy^2 + dz^2 - d\rho^2)}}{\rho} = d\sigma + d\psi\cos\omega,$$

unde concludimus

$$dx^2 + dy^2 + dz^2 = d\rho^2 + \rho\rho(d\sigma + d\psi\cos\omega)^2 = 2gLdt^2\left(2D + \frac{2}{\rho} - 2\int(Xdx + Ydy + Zdz)\right).$$

Statuamus brevitatis ergo $d\sigma + d\psi\cos\omega = d\varphi$, ut sit

$$d\rho^2 + \rho\rho d\varphi^2 = 2gLdt^2\left(2D + \frac{2}{\rho} - 2\int(Xdx + Ydy + Zdz)\right).$$

Tum vero ob $z = \rho\sin\sigma\sin\omega$ habebimus

$$\frac{x}{z} = \frac{\cos\sigma\cos\psi}{\sin\sigma\sin\omega} - \frac{\cos\omega\sin\psi}{\sin\omega} \quad \text{et} \quad \frac{y}{z} = \frac{\cos\sigma\sin\psi}{\sin\sigma\sin\omega} + \frac{\cos\omega\cos\psi}{\sin\omega},$$

unde per differentiationem ob $d\omega = \frac{d\psi\cos\sigma\sin\omega}{\sin\sigma}$, colligimus

$$\frac{zdx - xdz}{zz} = \frac{-d\varphi\cos\psi}{\sin^2\sigma\sin\omega} \quad \text{et} \quad \frac{zdy - ydz}{zz} = \frac{-d\varphi\sin\psi}{\sin^2\sigma\sin\omega},$$

hincque porro $zdx - xdz = -vvd\varphi\cos\psi\sin\omega$ et $zdy - ydz = -vvd\varphi\sin\psi\sin\omega$. Est vero ex aequationibus principalibus:

$$zddx - xddz = 2gLdt^2(Zx - Xz) \quad \text{et} \quad zddy - yddz = 2gLdt^2(Zy - Yz),$$

quarum illa per $2(zdx - xdz)$, haec vero per $2(zdy - ydz)$ multiplicata et integrata dabit

$$(zdx - xdz)^2 = v^4 d\varphi^2\cos^2\psi\sin^2\omega = 4gLdt^2\int vvd\varphi\cos\psi\sin\omega(Xz - Zx),$$
$$(zdy - ydz)^2 = v^4 d\varphi^2\sin^2\psi\sin^2\omega = 4gLdt^2\int vvd\varphi\sin\psi\sin\omega(Yz - Zy),$$

quibus additis prodit

$$v^4 d\varphi^2\sin^2\omega = 4gLdt^2\int v^3 d\varphi\sin\omega(\sin\sigma\sin\omega(X\cos\psi + Y\sin\psi) - Z\cos\sigma).$$

At si illae aequationes differentientur, indeque differentiale ipsius $v^4 d\varphi^2\sin^2\omega$ eliminetur, obtinebimus

$$vd\varphi d\psi\sin\omega = 2gLdt^2\sin\sigma(\sin\omega(Y\cos\psi - X\sin\psi) - Z\cos\omega),$$

ita ut sit

$$d\psi = \frac{2gLdt^2\sin\sigma}{vd\varphi}(Y\cos\psi - X\sin\psi - Z\cot\omega).$$

Ponamus brevitatis gratia

$$\int v^3 d\varphi\sin\omega(\sin\sigma\sin\omega(X\cos\psi + Y\sin\psi) - Z\cos\sigma) = S,$$

ut sit $v^4 d\varphi^2\sin^2\omega = 4gLdt^2(C + S)$, fietque

$$dv^2 = 4gLdt^2\left(D + \frac{1}{v} - \int(Xdx + Ydy + Zdz) - \frac{C - S}{vv\sin^2\omega}\right), \quad \text{seu}$$

$$dv^2(C + S) = v^4 d\varphi^2\sin^2\omega\left(D + \frac{1}{v} - \int(Xdx + Ydy + Zdz) - \frac{C - S}{vv\sin^2\omega}\right),$$

ac praeterea $$d\psi = \frac{v^3 d\varphi\sin\omega\sin\sigma}{2(C + S)}(\sin\omega(Y\cos\psi - X\sin\psi) - Z\cos\omega).$$

Cum igitur X, Y, Z sint quantitates valde parvae, erit etiam S quantitas minima, et anguli ψ et ω fere constantes, ita ut sit proxime $d\varphi = d\sigma$, accuratius autem $d\sigma = d\varphi - d\psi\cos\omega$. Denique vero erit

$$\frac{d\omega}{\sin^2\omega} = \frac{v^3 d\varphi\cos\sigma}{2(C + S)}(\sin\omega(Y\cos\psi - X\sin\psi) - Z\cos\omega),$$

et aequationis hujus

$$\frac{dv}{vv}\sqrt{(C + S)} = d\varphi\sin\omega\sqrt{\left(D + \frac{1}{v} - \frac{C - S}{vv\sin^2\omega} - \int(Xdx + Ydy + Zdz)\right)}$$

resolutio est instituenda ut ante docuimus.

170. **Coroll. 1.** Uti ω inclinatio orbitae et ψ longitudo nodi ascendentis vocari solet, ita angulus $☊JN = \sigma$ argumentum latitudinis et angulus φ longitudo in orbita appellatur, quae autem tantum ficta, cum tam linea nodorum quam inclinatio continuo mutetur.

171. **Coroll. 2.** Si vires exiguae ita fuerint comparatae, ut sit

$$\sin\omega\,(Y\cos\psi - X\sin\psi) - Z\cos\omega = 0,$$

tum ob $d\psi = 0$ et $d\omega = 0$, tam linea nodorum quam inclinatio nullam patitur mutationem, ideoque corpus N in eodem perpetuo plano feretur.

172. **Coroll. 3.** Cum autem angulus in plano AJB sumtus AJY vocetur corporis longitudo, erit $= \psi + \text{Ang.tang}\,(\text{tang}\,\sigma\cos\omega)$, tum vero latitudo corporis, quae est angulus YJN, est angulus cujus sinus est $\frac{z}{v} = \sin\sigma\sin\omega$.

173. **Scholion.** Haec methodus motum corporis ad planum fixum reducendi illi multum anteferenda videtur, qua ipsa corporis longitudo seu angulus AJY in calculum introducitur, quo pacto formulae satis intricatae redduntur. Hoc igitur incommodum hic maximam partem sustulimus, dum angulum σ, quo argumentum latitudinis denotatur, ac praeterea longitudinem in orbita seu angulum φ induximus, quoniam hoc modo formulae $zdx - xdz$ et $zdy - ydz$ tam commode exprimuntur, unde etiam fit

$$ydx - xdy = -vvd\varphi\cos\omega \quad \text{atque} \quad yddx - xddy = 2gLdt^2\,(Yx - Xy).$$

Haec ergo per $2\,(ydx - xdy)$ multiplicata et integrata dabit

$$(ydx - xdy)^2 = 4gLdt^2 \int vvd\varphi\cos\omega\,(Xy - Yx) = v^4 d\varphi^2\cos^2\omega,$$

quae etsi jam in praecedentibus contineatur, saepe ingentem usum praestat, uti in sequente problemate patebit. Hinc scilicet commode relatio inter dt et $d\varphi$ desumi poterit. Deinde etiam vis hujus methodi in hoc consistit, quod elementum temporis dt penitus e formulis integralibus exclusimus, quo deinceps commode ex calculo eliminari posset.

174. **Problema.** Si corpus M, cujus momenta inertiae respectu axium JA et JB sint aequalia, circa tertium axem JC utcunque gyretur, ac circa id corpus sphaericum N quomodocunque moveatur, hujus corporis N motum definire.

Solutio. (Fig. 183.) Plano axium JA et JB, quod quasi est corporis M planum aequatoris, pro plano fixo assumto, sit Maa momentum inertiae respectu axium JA et JB, at Mcc respectu axis JC. Pro motu ergo secundum problema praecedens definiendo habebimus ex § 128 has aequationes

$$ddx = \frac{-2g\,(M+N)\,xdt^2}{v^3}\left(1 + \frac{3\,(4aa+cc)}{2vv} - \frac{15\,(aaxx+aayy+cczz)}{2v^4}\right),$$

$$ddy = \frac{-2g\,(M+N)\,ydt^2}{v^3}\left(1 + \frac{3\,(4aa+cc)}{2vv} - \frac{15\,(aaxx+aayy+cczz)}{2v^4}\right),$$

$$ddz = \frac{-2g\,(M+N)\,zdt^2}{v^3}\left(1 + \frac{3\,(2aa+3cc)}{2vv} - \frac{15\,(aaxx+aayy+cczz)}{2v^4}\right),$$

quibus comparatis cum ante assumtis erit $L = M + N$ et

$$X = \frac{3x(4aa+cc)}{2v^5} - \frac{15x(aaxx+aayy+cczz)}{2v^7},$$

$$Y = \frac{3y(4aa+cc)}{2v^5} - \frac{15y(aaxx+aayy+cczz)}{2v^7},$$

$$Z = \frac{3z(2aa+3cc)}{2v^5} - \frac{15z(aaxx+aayy+cczz)}{2v^7},$$

hinc ob $xdx + ydy + zdz = vdv$, erit

$$Xdx + Ydy + Zdz = \frac{3(4aa+cc)(xdx+ydy)}{2v^5} + \frac{3(2aa+3cc)zdz}{2v^5} - \frac{15dv(aaxx+aayy+cczz)}{2v^6}$$

$$= \frac{3(4aa+cc)dv}{2v^4} - \frac{3(aa-cc)zdz}{v^5} - \frac{15aadv}{2v^4} + \frac{15(aa-cc)zzdv}{2v^6}.$$

Ergo $\int(Xdx + Ydy + Zdz) = \frac{(aa-cc)}{2v^3} - \frac{3(aa-cc)zz}{2v^5}$, hincque

$$dv^2 + vvd\varphi^2 = 4gLdt^2\left(D + \frac{1}{v} + \frac{(cc-aa)}{2v^3} - \frac{3(cc-aa)zz}{2v^5}\right).$$

Cum nunc ex § praecedente sit $yddx - xddy = 0$, erit

$$ydx - xdy = -vvd\varphi\cos\omega = -Edt\sqrt{4gL} \quad \text{et} \quad vvd\varphi^2 = \frac{4gLEEdt^2}{vv\cos^2\omega}, \quad \text{hincque}$$

$$dv^2 = 4gLdt^2\left(D + \frac{1}{v} + \frac{cc-aa}{2v^3} - \frac{3(cc-aa)zz}{2v^5} - \frac{EE}{vv\cos^2\omega}\right) \quad \text{et}$$

$$dv^2 = \frac{v^4d\varphi^2\cos^2\omega}{EE}\left(D + \frac{1}{v} + \frac{cc-aa}{2v^3} - \frac{3(cc-aa)zz}{2v^5} - \frac{EE}{vv\cos^2\omega}\right), \quad \text{seu}$$

$$\frac{Edv}{vv} = d\varphi\cos\omega\sqrt{\left(D + \frac{1}{v} + \frac{cc-aa}{2v^3} - \frac{3(cc-aa)\sin^2\sigma\sin^2\omega}{2v^3} - \frac{EE}{vv\cos^2\omega}\right)}$$

atque $2Edt\sqrt{gL} = vvd\varphi\cos\omega$. Deinde vero habemus

$$v^4d\varphi^2\cos^2\psi\sin^2\omega = 12gL(aa-cc)dt^2\int\frac{xzd\varphi\cos\psi\sin\omega}{v^3} \quad \text{et}$$

$$v^4d\varphi^2\sin^2\psi\sin^2\omega = 12gL(aa-cc)dt^2\int\frac{yzd\varphi\sin\psi\sin\omega}{v^3},$$

quibus additis fit

$$v^4d\varphi^2\sin^2\omega = 12gL(aa-cc)dt^2\int\frac{d\varphi\sin\sigma\cos\sigma\sin^2\omega}{v}.$$

Cum porro sit $v^4d\varphi^2\cos^2\omega = 4gLEEdt^2$, erit differentiando

$$2v^4d\varphi^2d\omega\sin\omega\cos\omega + \sin^2\omega d.(v^4d\varphi^2) = 12gL(aa-cc)dt^2\cdot\frac{d\varphi\sin\sigma\cos\sigma\sin^2\omega}{v}$$

$$\text{et} \quad -2v^4d\varphi^2d\omega\sin\omega\cos\omega + \cos^2\omega.d(v^4d\varphi^2) = 0,$$

unde concluditur

$$2v^4\, d\varphi^2\, d\omega \sin\omega \cos\omega = 12 gL\,(aa - cc)\, dt^2 \cdot \frac{d\varphi \sin\sigma \cos\sigma \sin^2\omega \cos^2\omega}{v}, \quad \text{seu}$$

$$d\varphi\, d\omega = \frac{6 gL\,(aa - cc)\, dt^2 \sin\sigma \cos\sigma \sin\omega \cos\omega}{v^5}, \quad \text{ideoque}$$

$$d\omega = \frac{3\,(aa - cc)\, d\varphi \sin\sigma \cos\sigma \sin\omega \cos^3\omega}{2EEv} \quad \text{et} \quad d\psi = \frac{d\omega \sin\sigma}{\cos\sigma \sin\omega} = \frac{3\,(aa - cc)\, d\varphi \sin^2\sigma \cos^3\omega}{2EEv}.$$

Initio igitur ob $aa - cc$ minimum, elementa ψ et ω ut constantia spectantur, et cum sit $d\varphi = d\sigma + d\psi \cos\omega$, differentialia $d\varphi$ et $d\sigma$ pro aequalibus habentur. Ponatur jam $EE = F\cos^2\omega$ et $(cc - aa)(1 - 3\sin^2\sigma \sin^2\omega) = G$, ut habeamus

$$\frac{Edv}{vv} = d\varphi \cos\omega \sqrt{\left(D + \frac{1}{v} - \frac{F}{vv} + \frac{G}{v^3}\right)}.$$

Ponatur nunc $v = \frac{p}{1 + q\cos s}$, fiatque $D + \frac{(1 \pm q)}{p} - F\left(\frac{1 \pm q}{p}\right)^2 + G\left(\frac{1 \pm q}{p}\right)^3 = 0$, ut sit

$$D + \frac{1}{p} - \frac{F(1 + qq)}{pp} + \frac{G(1 + 3qq)}{p^3} = 0 \quad \text{et} \quad 1 - \frac{2F}{p} + \frac{G(3 + qq)}{pp} = 0,$$

ubi cum G sit valde parvum, sit $F = \frac{f}{2} + u$, ut prodeat valor prope verus $p = f$, eritque

$$EE = \frac{1}{2} f\cos^2\omega + u\cos^2\omega = \text{Constanti}.$$

Sit ε valor medius inclinationis et $EE = \frac{1}{2} f\cos^2\varepsilon$, erit

$$u = \frac{f(\cos^2\varepsilon - \cos^2\omega)}{2\cos^2\omega}, \quad \text{atque} \quad 1 - \frac{f}{p} - \frac{(\cos^2\varepsilon - \cos^2\omega)}{\cos^2\omega} + \frac{G(3 + kk)}{ff} = 0, \quad \text{et hinc}$$

$$\frac{1}{p} = \frac{1}{f} - \frac{(\cos^2\varepsilon - \cos^2\omega)}{f\cos^2\omega} + \frac{G(3 + kk)}{f^3}.$$

Tum vero prior aequatio erit

$$D + \frac{1}{p} - \frac{f(1 + qq)}{2pp} - \frac{(1 + kk)(\cos^2\varepsilon - \cos^2\omega)}{2f\cos^2\omega} + \frac{G(1 + 3kk)}{f^3} = 0.$$

Sit constans $D = \frac{kk - 1}{2f}$, eritque

$$\frac{qq}{pp} = \frac{kk}{ff} - \frac{(1 + kk)(\cos^2\varepsilon - \cos^2\omega)}{ff\cos^2\omega} + \frac{2G(1 + 3kk)}{f^4}, \quad \text{ideoque}$$

$$p = f + \frac{f(\cos^2\varepsilon - \cos^2\omega)}{\cos^2\omega} - \frac{G(3 + kk)}{f} \quad \text{et} \quad qq = kk - \frac{(1 - kk)(\cos^2\varepsilon - \cos^2\omega)}{\cos^2\omega} + \frac{2G(1 - k^4)}{ff},$$

unde formula irrationalis abit in

$$\frac{q\sin s}{p} \sqrt{\left(\frac{f}{2} + \frac{f(\cos^2\varepsilon - \cos^2\omega)}{2\cos^2\omega} - \frac{G(3 + k\cos s)}{f}\right)},$$

ut ob $E = \frac{\cos\varepsilon \sqrt{f}}{\sqrt{2}}$ sit

$$\frac{dv}{vv} = \frac{qd\varphi \sin s \cos\omega}{p \cos\varepsilon} \sqrt{\left(1 + \frac{\cos^2\varepsilon - \cos^2\omega}{\cos^2\omega} - \frac{2G(3 + k\cos s)}{ff}\right)}, \quad \text{seu}$$

$$\frac{dv}{vv} = \frac{q\sin s}{p} d\varphi \sqrt{\left(1 - \frac{2G(3 + k\cos s)\cos^2\omega}{ff\cos^2\varepsilon}\right)} = \frac{q\sin s}{p}\left(d\varphi - \frac{Gd\varphi(3 + k\cos s)\cos^2\omega}{ff\cos^2\varepsilon}\right).$$

Per differentiationem autem obtinemus

$$\frac{dp}{pp} = \frac{3(cc-aa)\,d\varphi \sin\sigma\cos\sigma\sin^2\omega\,(1 - 2k\cos s + kk)}{f^3},$$

$$\frac{pdq - qdp}{pp} = \frac{3(cc-aa)\,d\varphi\sin\sigma\cos\sigma\sin^2\omega\,(\cos s - 2k + kk\cos s)}{f^3},$$

hincque concludimus

$$\frac{dv}{vv} = \frac{qds\sin s}{p} + \frac{3(cc-aa)\,d\varphi\sin\sigma\cos\sigma\sin^2\omega\,(1-kk)\sin^2 s}{f^3}$$

$$= \frac{qd\varphi\sin s}{p} - \frac{k(cc-aa)(1 - 3\sin^2\sigma\sin^2\omega)\,d\varphi\,(3 + k\cos s)\cos^2\omega\sin s}{2f^3\cos^2\varepsilon},$$

ita ut sit

$$d\varphi - ds = \frac{3(cc-aa)\,d\varphi\sin\sigma\cos\sigma\sin^2\omega\,(1-kk)\sin s}{ffk} + \frac{(cc-aa)(1 - 3\sin^2\sigma\sin^2\omega)(3 + k\cos s)\,d\varphi\cos^2\omega}{2ff\cos^2\varepsilon}.$$

Cum igitur in his terminis minimis liceat ponere $\omega = \varepsilon$, quae est inclinatio media, erit

$$d\varphi - ds = \frac{(cc-aa)(3 + k\cos s)d\varphi}{2ff} - \frac{3(cc-aa)(3 + k\cos s)\,d\varphi\sin^2\varepsilon\sin^2\sigma}{2ff} + \frac{3(cc-aa)(1-kk)\,d\varphi\sin^2\varepsilon\sin s\sin\sigma\cos\sigma}{ffk}$$

ubi statuere licet $d\varphi = ds = d\sigma$. Tum vero habetur

$$p = \frac{f\cos^2\varepsilon}{\cos^2\omega} - \frac{(cc-aa)(1 - 3\sin^2\varepsilon\sin^2\sigma)(3 + kk)}{2f},$$

$$qq = \frac{kk\cos^2\varepsilon}{\cos^2\omega} + 1 - \frac{\cos^2\varepsilon}{\cos^2\omega} + \frac{(cc-aa)(1 - 3\sin^2\varepsilon\sin^2\sigma)(1 - k^4)}{ff}$$

ac praeterea

$$d\psi = \frac{-3(cc-aa)(1 + k\cos s)\,d\varphi\cos\varepsilon\sin^2\sigma}{ff}, \qquad d\omega = \frac{-3(cc-aa)(1 + k\cos s)\,d\varphi\sin\varepsilon\cos\varepsilon\sin\sigma\cos\sigma}{ff},$$

eritque $d\varphi = d\sigma + d\psi\cos\varepsilon$, ac tandem pro tempore

$$dt\sqrt{2fgL} = \frac{vv\,d\varphi\cos\omega}{\cos\varepsilon} = \frac{pp\,d\varphi\cos\omega}{\cos\varepsilon\,(1 + q\cos s)^2},$$

quae formulae omnes in terminis minimis sine difficultate integrari possunt; postrema tantum formula majorem solertiam postulat. Ponamus enim ad abbreviandum $\frac{cc-aa}{ff} = n$ et evolutis productis sinuum et cosinuum adipiscemur

$$d\psi = -\tfrac{3}{2}n\,d\varphi\cos\varepsilon\,(1 + k\cos s - \cos 2\sigma - \tfrac{1}{2}k\cos(2\sigma - s) - \tfrac{1}{2}k\cos(2\sigma + s)),$$

$$d\omega = -\tfrac{3}{2}n\,d\varphi\sin\varepsilon\cos\varepsilon\,(\sin 2\sigma + \tfrac{1}{2}k\sin(2\sigma - s) + \tfrac{1}{2}k\sin(2\sigma + s)),$$

$$d\varphi - ds =$$

$$\tfrac{3}{2} nd\varphi\,(3 + k\cos s) - \tfrac{3}{4}\, nd\varphi \sin^2\varepsilon\,\Big(3 + k\cos s - 3\cos 2\sigma - \frac{(2-kk)}{2k}\cos(2\sigma - s) + \frac{2-3kk}{2k}\cos(2\sigma + s)\Big),$$

$$d\varphi - d\sigma = -\tfrac{3}{2} nd\varphi \cos^2\varepsilon\,\big(1 + k\cos s - \cos 2\sigma - \tfrac{1}{2}k\cos(2\sigma - s) - \tfrac{1}{2}k\cos(2\sigma + s)\big).$$

Hic igitur totum negotium pendet ab integratione hujusmodi formulae $\int d\varphi \cos(\mu s + \nu\sigma)$, ad quam accurate evolvendam ponamus brevitatis gratia

$$d\varphi = ds + \alpha d\varphi + P d\varphi \quad \text{et} \quad d\varphi = d\sigma + \beta d\varphi + Q d\varphi, \quad \text{ut sit}$$

$$\alpha = \tfrac{3}{2} n - \tfrac{9}{4} n \sin^2\varepsilon, \qquad \beta = -\tfrac{3}{2} n \cos^2\varepsilon,$$

$$P = \tfrac{1}{2} nk\cos s - \tfrac{3}{4}\, n \sin^2\varepsilon\,\Big(k\cos s - 3\cos 2\sigma - \frac{(2-kk)}{2k}\cos(2\sigma - s) + \frac{2-3kk}{2k}\cos(2\sigma + s)\Big) \quad \text{et}$$

$$Q = -\tfrac{3}{2} n\cos^2\varepsilon\,\big(k\cos s - \cos 2\sigma - \tfrac{1}{2}k\cos(2\sigma - s) - \tfrac{1}{2}k\cos(2\sigma + s)\big).$$

Quare cum hinc conficiatur $d\varphi = \dfrac{\mu ds + \nu d\sigma + d\varphi(\mu P + \nu Q)}{\mu + \nu - \alpha\mu - \beta\nu}$, erit

$$\int d\varphi \cos(\mu s + \nu\sigma) = \frac{\sin(\mu s + \nu\sigma)}{\mu + \nu - \alpha\mu - \beta\nu} + \int \frac{d\varphi(\mu P + \nu Q)\cos(\mu s + \nu\sigma)}{\mu + \nu - \alpha\mu - \beta\nu}.$$

Tum vero cum $\mu P + \nu Q$ habeat hujusmodi formam

$$A\cos s + B\cos 2\sigma + C\cos(2\sigma - s) + D\cos(2\sigma + s),$$

haec per $\cos(\mu s + \nu\sigma)$ multiplicata denuo in simplices cosinus evolvitur, quorum singuli praebent formulas similes integrandas. Qui etsi videntur ob parvitatem rejiciendi, tamen si in iis fiat $\mu + \nu = 0$, ob denominatorem $-\alpha\mu - \beta\nu$ minimum ad notabilem valorem exsurgere possunt.

175. **Coroll. 1.** Cum sit $d\omega = -\frac{3}{2} nd\varphi \sin\varepsilon\cos\varepsilon\,(\ldots)$ (vide ult. lin. pag. praec.) patet duobus casibus inclinationem orbitae nullam pati mutationem, altero quo $\varepsilon = 0$, seu corpus N in ipso plano aequatoris AJB movetur, altero quo $\varepsilon = 90^0$, seu corpus N in plano ad aequatorem perpendiculari fertur; atque hoc casu etiam linea nodorum est fixa. Ceteris ergo paribus inclinatio obnoxia erit maximae variationi, quando inclinatio ε est 45^0.

176. **Coroll. 2.** Pro motu lineae nodorum invenimus longitudinem nodi ascendentis

$$\psi = \text{Const.} - \tfrac{3}{2} n\varphi\cos\varepsilon - \tfrac{3}{2} n\cos\varepsilon\,\big(k\!\int\! d\varphi\cos s - \!\int\! d\varphi\cos 2\sigma - \tfrac{1}{2}k\!\int\! d\varphi\cos(2\sigma - s) - \tfrac{1}{2}k\!\int\! d\varphi\cos(2\sigma + s)\big).$$

Pro motu autem lineae absidum erit longitudo absidis imae

$$\varphi - s = \text{Const.} + \tfrac{3}{2} n\,(1 - \tfrac{3}{2}\sin^2\varepsilon)\,\varphi + \tfrac{1}{2} nk\,(1 - \tfrac{3}{2}\sin^2\varepsilon)\int d\varphi\cos s$$
$$+ \tfrac{3}{4} n\sin^2\varepsilon\,\Big(3\!\int\! d\varphi\cos 2\sigma + \frac{2-kk}{2k}\int d\varphi\cos(2\sigma - s) - \frac{(2-3kk)}{2k}\int d\varphi\cos(2\sigma + s)\Big)$$

et pro argumento latitudinis σ habemus $\varphi = \sigma + \psi\cos\varepsilon$.

177. **Coroll. 3.** Si partes integrales rejiciamus, innotescet vero proxime motus medius lineae nodorum quam lineae absidum, ac si $n = \frac{cc - aa}{ff}$ sit numerus positivus, linea nodorum regreditur, idque eo minus, quo major fuerit inclinatio. Linea autem absidum progreditur, quamdiu $\sin^2 \varepsilon < \frac{2}{3}$, seu $\varepsilon < 54^0 45'$; sin autem fuerit $\varepsilon > 54^0 45'$, etiam linea absidum regreditur.

178. **Coroll. 4.** Cum sit proxime $d\varphi = ds = d\sigma$, erunt integralium valores proxime

$$\int d\varphi \cos s = \sin s, \quad \int d\varphi \cos 2\sigma = \tfrac{1}{2} \sin 2\sigma, \quad \int d\varphi \cos (2\sigma - s) = \sin (2\sigma - s) \quad \text{et}$$

$$\int d\varphi \cos (2\sigma + s) = \tfrac{1}{3} \sin (2\sigma + s),$$

unde praeter motum medium utriusque lineae nodorum et absidum, anomaliae periodicae definiri possunt.

179. **Scholion.** Hae determinationes recte se habere sunt censendae, dummodo fractio $n = \frac{cc - aa}{ff}$ satis fuerit parva, ut termini quadrato nn affecti pro nihilo haberi queant. Sin autem eveniat, ut haec fractio non sit adeo parva, tum jam superiores formulae accuratius evolvi deberent, ut termini per nn multiplicati simul comprehenderentur; hoc autem modo in formulas nimis prolixas incideremus. Verum hinc statim ii termini excludi poterunt, qui nullius plane momenti videbuntur, iis tantum retentis, qui per integrationem insignes coëfficientes adipiscuntur, cujusmodi est $\cos(2\sigma - 2s)$, unde per integrationem oritur

$$\int d\varphi \cos (2\sigma - 2s) = \frac{\sin(2\sigma - 2s)}{2\alpha - 2\beta} = \frac{2 \sin (2\sigma - 2s)}{3n(2 - 3\sin^2\varepsilon + 2\cos^2\varepsilon)},$$

qui terminus etsi ex ordine per nn multiplicato nascitur, tamen ob denominatorem exiguum adjectionem per n multiplicatum elevatur. Deinde etiam si excentricitas k fuerit exigua, per integrationes ulterius productas anguli absoluti satis notabiles exsurgere possunt. Scilicet integratio $\int d\varphi \cos(2\sigma - s)$ ducit ad formam

$$\frac{\sin (2\sigma - s)}{1 + \alpha - 2\beta} + \frac{\int d\varphi (2Q - P) \cos (2\sigma - s)}{1 + \alpha - 2\beta},$$

at in $2Q - P$ continetur membrum

$$\tfrac{3}{2} nk \cos^2 \varepsilon \cos (2\sigma - s) - \frac{3n(2 - kk)}{8k} \sin^2 \varepsilon \cos (2\sigma - s),$$

quod per $\cos (2\sigma - s)$ multiplicatum praebet quantitatem constantem

$$\tfrac{3}{4} nk \cos^2 \varepsilon - \frac{3n(2 - kk)}{16k} \sin^2 \varepsilon,$$

ita ut inde oritur angulus absolutus

$$\left(\tfrac{3}{4} nk \cos^2 \varepsilon - \frac{3n(2 - kk)}{16k} \sin^2 \varepsilon\right) \varphi$$

ad motum medium adjiciendus. Simili modo ex formula

$$\int d\varphi \cos(2\sigma + s) = \frac{\sin(2\sigma + s)}{3 - \alpha - 2\beta} + \frac{\int d\varphi\,(2Q + P)\cos(2\sigma + s)}{3 - \alpha - 2\beta},$$

ob $2Q + P$ complectentem terminum $(\frac{3}{2} nk \cos^2\varepsilon - \frac{3n(2 - 3kk)}{8k}\sin^2\varepsilon)\cos(2\sigma + s)$, nascetur angulus absolutus $(\frac{1}{4} nk \cos^2\varepsilon - \frac{n(2 - 3kk)}{16k}\sin^2\varepsilon)\,\varphi$. Cum deinde in motu lineae absidum hi anguli denuo per $\frac{3n(2 - kk)}{8k}\sin^2\varepsilon$ et $\frac{-3n(2 - 3kk)}{8k}\sin^2\varepsilon$ multiplicari debeant, fieri potest, ut inde motus medius non afficiatur. Verum si hi termini alicujus sint momenti, etiam ipsas formulas principales accuratius evolvi oporteret, quod autem negotium hic suscipi non convenit, cum nondum satis constet, quibusnam casibus id utilitatem esset habiturum. Quod denique ad integrationem formulae

$$\int \frac{pp\,d\varphi \cos\omega}{\cos\varepsilon\,(1 + q\cos s)^2} = t\sqrt{2fgL}$$

attinet, in ea vires analyseos experiri oportet, ac tutissima quidem methodus videtur, postquam loco $d\varphi$ valor $ds + \alpha d\varphi + P d\varphi$ est positus, formulam $\frac{pp\,ds \cos\omega}{\cos\varepsilon\,(1 + q\cos s)^2}$ ita integrare, quasi p, q et ω essent constantes, tum vero invento integrali correctiones ex harum quantitatum variabilitate oriundas investigare. Atque haec de motu duorum corporum se mutuo attrahentium sufficere videntur, ex quo ad considerationem trium corporum progrediamur.

Caput VI.

De motu trium corporum sphaericorum, se mutuo attrahentium in genere.

180. **Problema.** (Fig. 185.) Si tria corpora sphaerica L, M, N, se mutuo attrahentia moveantur in eodem plano, eorum motum per calculum definire.

Solutio. Elapso tempore $= t$ versentur corpora in L, M, N in plano tabulae, in quo sumta recta fixa OV, ad quam eorum situs referatur, per puncta L, M, N agantur rectae $l\lambda$, $m\mu$, $n\nu$, ipsi OV parallelae, simulque ad eam perpendicula LP, MQ, NR. Quodsi jam longitudinem cujusque corporis ex altero spectati per angulum a recta OV in sensum Vv sumtum aestimemus, statuamus

longitudinem corporis M ex L spectati $lLM = \zeta$

longitudinem corporis N ex M spectati $mMN = \eta$

longitudinem corporis L ex N spectati $nNL = \vartheta$,

qui postremus angulus ϑ in figura duobus rectis major est intelligendus. Atque iidem anguli duobus rectis vel aucti vel minuti exhibebunt longitudinem corporum L, M, N ex M, N, L spectatorum. Ponamus nunc distantias $LM = x$, $MN = y$ et $NL = z$, erunt coordinatae

$$OQ = OP + x\cos\zeta, \qquad QM = PL + x\sin\zeta$$
$$OR = OQ + y\cos\eta, \qquad RN = QM + y\sin\eta$$
$$OP = OR + z\cos\vartheta, \qquad PL = RN + z\sin\vartheta$$

hincque colligimus

$$x\cos\zeta + y\cos\eta + z\cos\vartheta = 0 \quad \text{et} \quad x\sin\zeta + y\sin\eta + z\sin\vartheta = 0$$

ac porro

$$x\sin(\zeta - \vartheta) + y\sin(\eta - \vartheta) = 0, \qquad x\sin(\zeta - \eta) + z\sin(\vartheta - \eta) = 0,$$
$$y\sin(\eta - \zeta) + z\sin(\vartheta - \zeta) = 0,$$

ideoque $\quad x : y : z = \sin(\eta - \vartheta) : \sin(\vartheta - \zeta) : \sin(\zeta - \eta),$

unde relatio inter distantias et angulos ita commodissime exhibetur, ut sit

$$x = v\sin(\eta - \vartheta), \qquad y = v\sin(\vartheta - \zeta), \qquad z = v\sin(\zeta - \eta),$$

ubi v denotat diametrum circuli triangulo LMN circumscripti. Si jam massae corporum litteris cognominibus L, M, N exprimantur, corpus L a reliquis sollicitatur

$$\text{sec. } OP \text{ vi} = \frac{LM\cos\zeta}{xx} - \frac{LN\cos\vartheta}{zz} \qquad \text{et} \qquad \text{sec. } PL \text{ vi} = \frac{LM\sin\zeta}{xx} - \frac{LN\sin\vartheta}{zz},$$

corpus vero M a reliquis sollicitatur

$$\text{sec. } OQ \text{ vi} = \frac{MN\cos\eta}{yy} - \frac{LM\cos\zeta}{xx} \qquad \text{et} \qquad \text{sec. } QM \text{ vi} = \frac{MN\sin\eta}{yy} - \frac{LM\sin\zeta}{xx}$$

et corpus N a reliquis sollicitatur

$$\text{sec. } OR \text{ vi} = \frac{LN\cos\vartheta}{zz} - \frac{MN\cos\eta}{yy} \qquad \text{et} \qquad \text{sec. } RN \text{ vi} = \frac{LN\sin\vartheta}{zz} - \frac{MN\sin\eta}{yy},$$

unde sequentes aequationes adipiscimur

$$dd\,.\,OP = 2gdt^2\left(\frac{M\cos\zeta}{xx} - \frac{N\cos\vartheta}{zz}\right), \qquad dd\,.\,PL = 2gdt^2\left(\frac{M\sin\zeta}{xx} - \frac{N\sin\vartheta}{zz}\right),$$
$$dd\,.\,OQ = 2gdt^2\left(\frac{N\cos\eta}{yy} - \frac{L\cos\zeta}{xx}\right), \qquad dd\,.\,QM = 2gdt^2\left(\frac{N\sin\eta}{yy} - \frac{L\sin\zeta}{xx}\right),$$
$$dd\,.\,OR = 2gdt^2\left(\frac{L\cos\vartheta}{zz} - \frac{M\cos\eta}{yy}\right), \qquad dd\,.\,RN = 2gdt^2\left(\frac{L\sin\vartheta}{zz} - \frac{M\sin\eta}{yy}\right),$$

ex quibus colligimus sequentes

$$dd.x\cos\zeta = 2gdt^2\left(-\frac{(L+M)\cos\zeta}{xx} + \frac{N\cos\eta}{yy} + \frac{N\cos\vartheta}{zz}\right), \quad dd.x\sin\zeta = 2gdt^2\left(-\frac{(L+M)\sin\zeta}{xx} + \frac{N\sin\eta}{yy} + \frac{N\sin\vartheta}{zz}\right)$$
$$dd.y\cos\eta = 2gdt^2\left(-\frac{(M+N)\cos\eta}{yy} + \frac{L\cos\vartheta}{zz} + \frac{L\cos\zeta}{xx}\right), \quad dd.y\sin\eta = 2gdt^2\left(-\frac{(M+N)\sin\eta}{yy} + \frac{L\sin\vartheta}{zz} + \frac{L\sin\zeta}{xx}\right)$$
$$dd.z\cos\vartheta = 2gdt^2\left(-\frac{(L+N)\cos\vartheta}{zz} + \frac{M\cos\zeta}{xx} + \frac{M\cos\eta}{yy}\right), \quad dd.z\sin\vartheta = 2gdt^2\left(-\frac{(L+N)\sin\vartheta}{zz} + \frac{M\sin\zeta}{xx} + \frac{M\sin\eta}{yy}\right)$$

quae porro transformantur in has

$$\text{I.}\quad 2dxd\zeta + xdd\zeta = 2gdt^2\left(\frac{N\sin(\eta-\zeta)}{yy} + \frac{N\sin(\vartheta-\zeta)}{zz}\right),$$

$$\text{II.}\quad ddx - xd\zeta^2 = 2gdt^2\left(-\frac{(L+M)}{xx} + \frac{N\cos(\eta-\zeta)}{yy} + \frac{N\cos(\vartheta-\zeta)}{zz}\right),$$

$$\text{III.}\quad 2dyd\eta + ydd\eta = 2gdt^2\left(\frac{L\sin(\vartheta-\eta)}{zz} + \frac{L\sin(\zeta-\eta)}{xx}\right),$$

$$\text{IV.}\quad ddy - yd\eta^2 = 2gdt^2\left(-\frac{(M+N)}{yy} + \frac{L\cos(\vartheta-\eta)}{zz} + \frac{L\cos(\zeta-\eta)}{xx}\right),$$

$$\text{V.}\quad 2dzd\vartheta + zdd\vartheta = 2gdt^2\left(\frac{M\sin(\zeta-\vartheta)}{xx} + \frac{M\sin(\eta-\vartheta)}{yy}\right),$$

$$\text{VI.}\quad ddz - zd\vartheta^2 = 2gdt^2\left(-\frac{(L+N)}{zz} + \frac{M\cos(\zeta-\vartheta)}{xx} + \frac{M\cos(\eta-\vartheta)}{yy}\right),$$

Ex harum aequationum I, III et V colligimus hanc integralem

$$LMxxd\zeta + MNyy\,d\eta + LNzz\,d\vartheta = Cdt,$$

deinde ex I et II deducimus

$$d.(dx^2 + xxd\zeta^2) = 4gdt^2\left(-\frac{(L+M)\,dx}{xx} + \frac{N(dx\cos(\eta-\zeta) + xd\zeta\sin(\eta-\zeta))}{yy} + \frac{N(dx\cos(\vartheta-\zeta) + xd\zeta\sin(\vartheta-\zeta))}{zz}\right),$$

quae ita repraesentetur

$$\frac{d.(dx^2 + xxd\zeta^2)}{4gNdt^2} = -\frac{(L+M)\,dx}{Nxx} + \frac{d.x\cos(\eta-\zeta) + xd\eta\sin(\eta-\zeta)}{yy} + \frac{d.x\cos(\vartheta-\zeta) + xd\vartheta\sin(\vartheta-\zeta)}{zz},$$

similesque ex reliquis ortae erunt

$$\frac{d.(dy^2 + yyd\eta^2)}{4gLdt^2} = \frac{-(M+N)\,dy}{Lyy} + \frac{d.y\cos(\vartheta-\eta) + yd\vartheta\sin(\vartheta-\eta)}{zz} + \frac{d.y\cos(\zeta-\eta) + yd\zeta\sin(\zeta-\eta)}{xx},$$

$$\frac{d.(dz^2 + zzd\vartheta^2)}{4gMdt^2} = \frac{-(L+N)\,dz}{Mzz} + \frac{d.z\cos(\zeta-\vartheta) + zd\zeta\sin(\zeta-\vartheta)}{xx} + \frac{d.z\cos(\eta-\vartheta) + zd\eta\sin(\eta-\vartheta)}{yy}.$$

Addantur hae tres aequationes, et cum sit

$$x\sin(\eta-\zeta) + z\sin(\eta-\vartheta) = 0, \qquad x\sin(\vartheta-\zeta) + y\sin(\vartheta-\eta) = 0,$$
$$y\sin(\zeta-\eta) + z\sin(\zeta-\vartheta) = 0,$$

summa erit

$$-\frac{(L+M)\,dx}{Nxx} - \frac{(M+N)\,dy}{Lyy} - \frac{(L+N)\,dz}{Mzz} + \frac{d(x\cos(\eta-\zeta) + z\cos(\eta-\vartheta))}{yy} + \frac{d(x\cos(\vartheta-\zeta) + y\cos(\vartheta-\eta))}{zz}$$
$$+ \frac{d(y\cos(\zeta-\eta) + z\cos(\zeta-\vartheta))}{xx}.$$

Ex aequationibus $x\cos\zeta + y\cos\eta + z\cos\vartheta = 0$ et $x\sin\zeta + y\sin\eta + z\sin\vartheta = 0$ colligimus

$$x\cos(\vartheta-\zeta) + y\cos(\vartheta-\eta) + z = 0, \qquad x\cos(\eta-\zeta) + z\cos(\eta-\vartheta) + y = 0,$$
$$y\cos(\zeta-\eta) + z\cos(\zeta-\vartheta) + x = 0,$$

quibus valoribus inductis consequimur

$$\frac{d.(dx^2+xx\,d\zeta^2)}{4gN\,dt^2}+\frac{d.(dy^2+yy\,d\eta^2)}{4gL\,dt^2}+\frac{d.(dz^2+zz\,d\vartheta^2)}{4gM\,dt^2}=\frac{-(L+M+N)\,dx}{Nxx}-\frac{(L+M+N)\,dy}{Lyy}-\frac{(L+M+N)\,dz}{Mzz}$$

hincque integrando

$$\frac{dx^2+xx\,d\zeta^2}{N}+\frac{dy^2+yy\,d\eta^2}{L}+\frac{dz^2+zz\,d\vartheta^2}{M}=4g\,(L+M+N)\,dt^2\,\left(D+\frac{1}{Nx}+\frac{1}{Ly}+\frac{1}{Mz}\right),$$

$$LM\,(dx^2+xx\,d\zeta^2)+MN\,(dy^2+yy\,d\eta^2)+LN\,(dz^2+zz\,d\vartheta^2)=$$

$$4g\,(L+M+N)\,dt^2\,\left(E+\frac{LM}{x}+\frac{MN}{y}+\frac{LN}{z}\right),$$

ita ut jam habeamus duas aequationes integrales. Praeterea autem notasse convenit esse

$$LM\,(x\,ddx+dx^2)+MN\,(y\,ddy+dy^2)+LN\,(z\,ddz+dz^2)=$$

$$2g\,(L+M+N)\,dt^2\,\left(2E+\frac{LM}{x}+\frac{MN}{y}+\frac{LN}{z}\right),$$

etiamsi hinc nulla via ad novam integrationem aperiatur. Cum igitur septem habeamus quantitates scilicet tres distantias x, y, z, tres angulos ζ, η, ϑ et tempus t, quarum relationem mutuam definiri oportet, ad hoc opus est sex aequationibus, ad quarum numerum complendum habemus primo has duas aequationes finitas

$$\text{I.}\quad x\cos\zeta+y\cos\eta+z\cos\vartheta=0,\qquad \text{II.}\quad x\sin\zeta+y\sin\eta+z\sin\vartheta=0,$$

deinde binas aequationes jam per integrationem erutas

$$\text{III.}\quad LMxx\,d\zeta+MNyy\,d\eta+LNzz\,d\vartheta=C\,dt \qquad \text{et}$$

$$\text{IV.}\quad LM\,(dx^2+xx\,d\zeta^2)+MN\,(dy^2+yy\,d\eta^2)+LN\,(dz^2+zz\,d\vartheta^2)=$$

$$4g\,(L+M+N)\,dt^2\,\left(E+\frac{LM}{x}+\frac{MN}{y}+\frac{LN}{z}\right).$$

Loco duarum reliquarum binae trium sequentium commodissime accipientur

$$\text{V.}\quad 2\,dx\,d\zeta+x\,dd\zeta=2gN\,dt^2\left(\frac{\sin(\eta-\zeta)}{yy}+\frac{\sin(\vartheta-\zeta)}{zz}\right),$$

$$\text{VI.}\quad 2\,dy\,d\eta+y\,dd\eta=2gL\,dt^2\left(\frac{\sin(\vartheta-\eta)}{zz}+\frac{\sin(\zeta-\eta)}{xx}\right),$$

$$\text{VII.}\quad 2\,dz\,d\vartheta+z\,dd\vartheta=2gM\,dt^2\left(\frac{\sin(\zeta-\vartheta)}{xx}+\frac{\sin(\eta-\vartheta)}{yy}\right),$$

quarum resolutio hoc modo tentanda videtur. Multiplicentur hae tres postremae aequationes seorsim per certas formulas differentiales, ita ut membra posteriora fiant integrabilia seorsim, priorum autem summa talis efficiatur. Ob

$$x:y:z=\sin(\eta-\vartheta):\sin(\vartheta-\zeta):\sin(\zeta-\eta),$$

prior conditio impletur si multiplicetur

$$\text{aequatio V. per}\quad \frac{yz\sin(\vartheta-\zeta)\sin(\zeta-\eta)}{\sin^3(\vartheta-\zeta)-\sin^3(\zeta-\eta)}\,dP,$$

$$\text{aequatio VI. per}\quad \frac{xz\sin(\zeta-\eta)\sin(\eta-\vartheta)}{\sin^3(\zeta-\eta)-\sin^3(\eta-\vartheta)}\,dQ,$$

$$\text{aequatio VII. per}\quad \frac{xy\sin(\eta-\vartheta)\sin(\vartheta-\zeta)}{\sin^3(\eta-\vartheta)-\sin^3(\vartheta-\zeta)}\,dR;$$

enim integralia posteriorum membrorum fiunt

$$2gNP\,dt^2,\qquad 2gLQ\,dt^2,\qquad 2gMR\,dt^2;$$

ergo, ut priorum membrorum aggregatum reddatur integrabile, quem in finem idoneos functionum P, Q, R investigari convenit. Verum hic calculi subsidiis destituti istud nego- deserere cogimur.

181. **Coroll. 1.** Posito $\sin(\eta-\vartheta)=\frac{x}{\varrho}$, $\sin(\vartheta-\zeta)=\frac{y}{\varrho}$, $\sin(\zeta-\eta)=\frac{z}{\varrho}$, aequationes V, VI, VII. in has abeunt formas

$$\text{V.}\quad d.(xx\,d\zeta)=\frac{2gNx\,dt^2}{\varrho}\left(\frac{y}{zz}-\frac{z}{yy}\right),$$

$$\text{VI.}\quad d.(yy\,d\eta)=\frac{2gLy\,dt^2}{\varrho}\left(\frac{z}{xx}-\frac{x}{zz}\right),$$

$$\text{VII.}\quad d.(zz\,d\vartheta)=\frac{2gMz\,dt^2}{\varrho}\left(\frac{x}{yy}-\frac{y}{xx}\right),$$

ita per x, y, z determinatur, ut sit

$$\varrho=\frac{2xyz}{\sqrt{(2xxyy+2xxzz+2yyzz-x^4-y^4-z^4)}}.$$

182. **Coroll. 2.** Si ex illis aequationibus eliminemus dt^2, obtinebimus

$$\frac{2g\,dt^2}{\varrho}=\frac{yyzz\,d.(xx\,d\zeta)}{Nx\,(y^3-z^3)}=\frac{xxzz\,d.(yy\,d\eta)}{Ly\,(z^3-x^3)}=\frac{xxyy\,d.(zz\,d\vartheta)}{Mz\,(x^3-y^3)}.$$

etiam in plures alias formas has aequationes transfundere licet, neque tamen methodus patet hinc novam aequationem integralem eliciendi.

183. **Scholion 1.** Hoc igitur problema, cui vera determinatio omnium motuum coelestium innititur, vires analyseos superat, etiamsi corpora se mutuo attrahentia sphaerica et in eodem plano moveri assumsimus; quae ergo conditiones, si secus se haberent, atque imprimis si numerus corporum ternarium excederet, multo minus de solutione cogitare liceret; ex quo intelligitur in subsidium Astronomiae ingentem analyseos promotionem desiderari. Neque etiam in genere ulla via ad approximationes patet, quibus uti non licet, nisi vel unum trium corporum sit valde parvum, vel vis inde ad motum reliquorum perturbandum nata fuerit vehementer exigua. Si enim corpus N evanescat, nostrae aequationes tantum ad has binas redeunt

$$LMxx\,d\zeta=C\,dt\qquad\text{et}\qquad LM(dx^2+xx\,d\zeta^2)=4g(L+M)\,dt^2\left(E+\frac{LM}{x}\right),$$

quibus motus duorum corporum continetur, unde si massa N sit valde parva, hinc idoneae approximationes peti poterunt. Deinde si corpus N sit infinite remotum, ut distantiae y et z fiant aequationum differentio-differentialium primo expositarum binae priores jam totum negotium abeuntes in has formas:

$$2dx d\zeta + x dd\zeta = 0 \quad \text{et} \quad ddx - xd\zeta^2 = \frac{-2g(L+M)\,dt^2}{xx},$$

ita ut reliquas ne in computum quidem duci necesse sit, qui casus ex posterioribus aequationibus minus perspicitur, cum ibi reliquae quantitates praeter necessitatem calculo sint immixtae. in mundo ejusmodi casus existeret, ut trium corporum se mutuo attrahentium neque unius massa prae reliquis valde parva, neque unius distantia a reliquis vehementer magna, fateri cogimur motum nobis fore imperscrutabilem: verum commode in mundo usu venit, ut hujusmodi casus nobis nusquam deprehendatur, qua in re nostrae imbecillitati non parum consultum videtur. obrem contenti simus in methodum inquisivisse, cujus beneficio proxime saltem motum trium corporum determinare valeamus, quando inter terna corpora se invicem attrahentia unum reperiatur cujus vis in reliqua sive ob massae parvitatem, sive ob ejus enormem distantiam, quasi evanescat quippe qui solus casus relinquitur, in quo vires nostras experiri liceat.

184. **Scholion 2.** Cum igitur tam mundus alios motus non offerat, quam analysis ad illos investigandos non sit apta, nisi qui non multum a ratione motus in sectione conica recedant, omnem operam in inventione aberrationum ab hac motus lege collocari conveniet. Hanc ob rem motum regularem vocabimus, qui leges motus, quibus duo tantum corpora sphaerica se mutuo attrahentia sunt inventa, perfecte sequitur, cujusmodi motus, etiamsi forte nusquam in mundo locum habeat, tamen, quoniam discrimen nusquam est valde magnum, aberrationes seu perturbationes motus regularis per approximationes definire conabimur. In proposito igitur problemate motum trium corporum L, M, N ita comparatum assumamus, ut bina M et N respectu tertii L motu fere regulari revolvantur, unde hoc commodi consequimur, ut dum perturbationes alterius definire studemus, alterius motum tanquam regularem spectare queamus; cum enim perturbationes ab hoc in illo productae per se sint valde parvae, sive hoc posterius regulariter moveatur, sive parumper a regula recedat, nullum discrimen in perturbatione illius orietur. Ita quando in perturbationes motus lunae a sole oriundas inquirere volumus, motum solis respectu terrae tanquam regularem spectabimus; vicissim, si errores in motu terrae ab actione lunae nati definiri debeant, qui terra ad quietem redacta in motum solis transferuntur, motum lunae tanquam regularem spectare licebit. Cum igitur propositis tribus corporibus unum semper in quiete considerari possit, problema ita tractabimus, ut binorum reliquorum unum motu regulari ferri censeatur, pro alteroque tantum perturbationes investigentur. Quod si praestiterimus, non amplius difficile erit, problemati pro corporibus quotcunque propositis satisfacere, quia enim perturbationes satis sunt exiguae, quantae a singulis seorsim producantur, assignavisse sufficiat, quae deinceps conjunctae omnes perturbationes ab omnibus simul ortas exhibebunt.

185. **Problema.** (Fig. 186.) Si corpus N circa corpus L, quod in quiete spectamus, motu regulari feratur, tum vero in eodem plano corpus M circa L ita moveatur, ut ejus motus ab actione corporis N perturbetur, hujus motus perturbationes assignare.

Solutio. Cum hic ad motum respectivum attendamus, corpore L in quiete spectato, ductis LM, LN et MN, attractio mutua corporum L et M est $=\frac{L.M}{LM^2}$, corporum L et $N=\frac{L.N}{LN^2}$, que corporum M et $N=\frac{M.N}{MN^2}$. Cum nunc corpus L sollicitetur secundum LM vi $=\frac{L.M}{LM^2}$, et secundum LN vi $=\frac{L.N}{LN^2}$, hae vires in sensum oppositum et in ratione massarum mutatae binis utrisque corporibus applicari debent. Ductis ergo rectis MT et NV ipsis NL et ML parallelis, corpus M praeter vires secundum $ML=\frac{L.M}{LM^2}$ et secundum $MN=\frac{M.N}{MN^2}$ sollicitari censendum est viribus secundum $ML=\frac{M.M}{LM^2}$ et secundum $MT=\frac{M.N}{LN^2}$; at corpus N praeter vires secundum $NL=\frac{L.N}{LN^2}$ et secundum $NM=\frac{M.N}{MN^2}$, a viribus secundum $NL=\frac{N.N}{LN^2}$ et secundum $NV=\frac{M.N}{LM^2}$. Sumtis nunc duabus directionibus fixis altera LA, altera ad hanc normali, corpus M sollicitabitur

$$\text{sec. } LQ \text{ vi } =\frac{-M(L+M)\,LQ}{LM^3}+\frac{M.N.MS}{MN^3}-\frac{M.N.LR}{LN^3},$$

$$\text{sec. } QM \text{ vi } =\frac{-M(L+M)\,QM}{LM^3}+\frac{M.N.SN}{MN^3}-\frac{M.N.RN}{LN^3}.$$

Corpus vero N sollicitabitur

$$\text{sec. } LR \text{ vi } =\frac{-N(L+N)\,LR}{LN^3}-\frac{M.N.MS}{MN^3}-\frac{M.N.LQ}{LM^3},$$

$$\text{sec. } RN \text{ vi } =\frac{-N(L+N)\,RN}{LN^3}-\frac{M.N.SN}{MN^3}-\frac{M.N.QM}{LM^3}.$$

Ponamus jam coordinatas pro corpore M

$$LQ=x,\quad QM=y,\quad LM=\sqrt{(xx+yy)}=v,$$

pro corpore N vero $\quad LR=\mathfrak{x},\quad RN=\mathfrak{y},\quad LN=\sqrt{(\mathfrak{x}\mathfrak{x}+\mathfrak{y}\mathfrak{y})}=\mathfrak{v},$

tum $MN=\sqrt{((\mathfrak{x}-x)^2+(\mathfrak{y}-y)^2)}=w$, et aequationes differentio-differentiales motum utriusque corporis exprimentes, posito elemento temporis dt constante, erunt

$$\text{I.}\quad ddx=2gdt^2\left(\frac{-(L+M)\,x}{v^3}+\frac{N(\mathfrak{x}-x)}{w^3}-\frac{N\mathfrak{x}}{\mathfrak{v}^3}\right),$$

$$\text{II.}\quad ddy=2gdt^2\left(\frac{-(L+M)\,y}{v^3}+\frac{N(\mathfrak{y}-y)}{w^3}-\frac{N\mathfrak{y}}{\mathfrak{v}^3}\right),$$

$$\text{III.}\quad dd\mathfrak{x}=2gdt^2\left(\frac{-(L+N)\,\mathfrak{x}}{\mathfrak{v}^3}-\frac{M(\mathfrak{x}-x)}{w^3}-\frac{Mx}{v^3}\right),$$

$$\text{IV.}\quad dd\mathfrak{y}=2gdt^2\left(\frac{-(L+N)\,\mathfrak{y}}{\mathfrak{v}^3}-\frac{M(\mathfrak{y}-y)}{w^3}-\frac{My}{v^3}\right).$$

Cum autem motus corporis M non adeo perturbari sumatur, hypothesis nostra exigit, ut termini $\frac{N}{ww}$ et $\frac{N}{\mathfrak{v}\mathfrak{v}}$ sint prae $\frac{L+M}{vv}$ valde parvi, atque eodem jure termini $\frac{M}{ww}$ et $\frac{M}{vv}$ prae $\frac{L+N}{\mathfrak{v}\mathfrak{v}}$ valde exigui esse debent; quia alioquin determinatio motus vires calculi superaret.

Cum igitur motus corporis N pro cognito habeatur, quantitates $\mathfrak{x}$, $\mathfrak{y}$ et $\mathfrak{v}$ tanquam functiones cognitae temporis t spectari possunt, sicque tantum duae aequationes priores relinquuntur, ex quibus colligimus

$$y\,ddx - x\,ddy = 2g\,dt^2\left(\frac{N(\mathfrak{x}y - x\mathfrak{y})}{w^3} - \frac{N(\mathfrak{x}y - x\mathfrak{y})}{\mathfrak{v}^3}\right), \quad \text{seu} \quad = 2gN(\mathfrak{x}y - x\mathfrak{y})\,dt^2\left(\frac{1}{w^3} - \frac{1}{\mathfrak{v}^3}\right)$$

$$\text{et} \quad 2\,dx\,ddx + 2\,dy\,ddy = 4g\,dt^2\left(\frac{-(L+M)\,dv}{vv} - \frac{Nv\,dv}{w^3} + N(\mathfrak{x}\,dx + \mathfrak{y}\,dy)\left(\frac{1}{w^3} - \frac{1}{\mathfrak{v}^3}\right)\right).$$

Ponamus nunc pro motu corporis M angulum $ALM = \varphi$, distantia existente $LM = v$, ut sit $x = v\cos\varphi$ et $y = v\sin\varphi$, hincque $y\,dx - x\,dy = -vv\,d\varphi$ et $dx^2 + dy^2 = dv^2 + vv\,d\varphi^2$. Similiter pro motu corporis N statuatur distantia $LN = u$, quae hactenus erat $= \mathfrak{v}$, et angulus $ALN = \vartheta$, ut sit $\mathfrak{x} = u\cos\vartheta$ et $\mathfrak{y} = u\sin\vartheta$, hincque

$$w = \sqrt{((u\cos\vartheta - v\cos\varphi)^2 + (u\sin\vartheta - v\sin\varphi)^2)} = \sqrt{(uu - 2uv\cos(\varphi - \vartheta) + vv)}$$

et $\quad \mathfrak{x}y - x\mathfrak{y} = uv\sin(\varphi - \vartheta)$, atque $\mathfrak{x}\,dx + \mathfrak{y}\,dy = u\,dv\cos(\varphi - \vartheta) - uv\,d\varphi\sin(\varphi - \vartheta)$.

Unde nostrae aequationes erunt

$$d.(vv\,d\varphi) = -2gNuv\,dt^2\sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

$$d.(dv^2 + vv\,d\varphi^2) = 4g\,dt^2\left(\frac{-(L+M)\,dv}{vv} - \frac{Nv\,dv}{w^3} + N(u\,dv\cos(\varphi - \vartheta) - uv\,d\varphi\sin(\varphi - \vartheta))\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right),$$

seu si differentialia secundi gradus non reformidemus,

$$2\,dv\,d\varphi + v\,dd\varphi = -2gNu\,dt^2\sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right) \quad \text{et}$$

$$ddv - v\,d\varphi^2 = -2g(L+M)\frac{dt^2}{vv} - 2gN\,dt^2\left(\frac{v}{w^3} - u\cos(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right),$$

ubi u et ϑ tanquam quantitates per t datae sunt spectandae, terminique per N affecti tanquam valde parvi.

Verum illae aequationes ad integrationem magis sunt praeparatae, et posterior ob

$$w\,dw = v\,dv + u\,du - u\,dv\cos(\varphi - \vartheta) - v\,du\cos(\varphi - \vartheta) + uv(d\varphi - d\vartheta)\sin(\varphi - \vartheta)$$

transit in hanc formam

$$d.(dv^2 + vv\,d\varphi^2) = -4g(L+M)\,dt^2\frac{dv}{vv} - 4gN\,dt^2\left(\frac{dv\cos(\varphi - \vartheta) - v\,d\varphi\sin(\varphi - \vartheta)}{uu}\right)$$

$$+ 4gN\,dt^2\left(\frac{u\,du - v\,du\cos(\varphi - \vartheta) - uv\,d\vartheta\sin(\varphi - \vartheta) - w\,dw}{w^3}\right),$$

unde integrando quatenus licet obtinemus

$$dv^2 + vv\,d\varphi^2 = 4g(L+M)\,dt^2\left(D + \frac{1}{v}\right) - 4gN\,dt^2\left(\frac{v\cos(\varphi - \vartheta)}{uu} - \int\frac{v\,d\vartheta\sin(\varphi - \vartheta)}{uu} + 2\int\frac{v\,du\cos(\varphi - \vartheta)}{u^3}\right)$$

$$+ 4gN\,dt^2\left(\frac{1}{w} + \int\frac{du\,(u + v\cos(\varphi - \vartheta))}{w^3} - \int\frac{uv\,d\vartheta\sin(\varphi - \vartheta)}{w^3}\right),$$

sive
$$dv^2 + vv\,d\varphi^2 = 4g(L+M)\,dt^2\left(D + \frac{1}{v}\right) + 4gN\,dt^2\left(\frac{1}{w} - \frac{v\cos(\varphi - \vartheta)}{uu}\right)$$

$$+ 4gN\,dt^2\int du\left(\frac{u - v\cos(\varphi - \vartheta)}{w^3} - \frac{2v\cos(\varphi - \vartheta)}{u^3}\right)$$

$$- 4gN\,dt^2\int uv\,d\vartheta\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\sin(\varphi - \vartheta).$$

vero aequatio per $2vvd\varphi$ multiplicata et integrata dat

$$v^4 d\varphi^2 = 4g(L+M)Cdt^2 - 4gNdt^2 \int uv^3 d\varphi \left(\frac{1}{w^3} - \frac{1}{u^3}\right) \sin(\varphi - \vartheta).$$

Ponamus brevitatis gratia

$$\int uv^3 d\varphi \left(\frac{1}{w^3} - \frac{1}{u^3}\right) \sin(\varphi - \vartheta) = P, \qquad \int uv\, d\vartheta \left(\frac{1}{w^3} - \frac{1}{u^3}\right) \sin(\varphi - \vartheta) = Q,$$

$$\int du \left(\frac{u - v\cos(\varphi - \vartheta)}{w^3} - \frac{2v\cos(\varphi - \vartheta)}{u^3}\right) = R,$$

ut habeamus has aequationes

$$v^4 d\varphi^2 = 4gdt^2 (C(L+M) - NP)$$

$$dv^2 + vvd\varphi^2 = 4gdt^2 \left(D(L+M) + \frac{L+M}{v} + \frac{N}{w} - \frac{Nv\cos(\varphi - \vartheta)}{uu} + NR - NQ\right),$$

unde eliminando $4gdt^2$ nanciscimur

$$dv^2\left(C(L+M) - NP\right) = v^4 d\varphi^2 \left(D(L+M) + \frac{L+M}{v} + \frac{N}{w} - \frac{Nv\cos(\varphi - \vartheta)}{uu} - NQ + NR - \frac{C(L+M)}{vv} + \frac{NP}{vv}\right).$$

Statuamus porro $\frac{N}{L+M} = n$, fietque

$$\frac{dv\sqrt{(C - nP)}}{vv} = d\varphi \sqrt{\left(D + \frac{1}{v} + \frac{n}{w} - \frac{nv\cos(\varphi - \vartheta)}{uu} - nQ + nR - \frac{C}{vv} + \frac{nP}{vv}\right)}$$

et $\qquad vvd\varphi = 2dt\sqrt{g(L+M)(C - nP)},$

ubi termini littera n affecti ut minimi spectantur. Illa autem aequatio etiam hoc modo exhiberi potest

$$\frac{dv\sqrt{(C - nP)}}{vv} = d\varphi \sqrt{\left(D + \frac{1}{v} - \frac{C}{vv} - 2n\int \frac{Pdv}{v^3} - n\int \frac{vdv}{w^3} + n\int udv \cos(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right)}.$$

Ponatur $v = \frac{p}{1 + q\cos s}$ et $C = \frac{f}{2}$ atque $D = \frac{kk - 1}{2f}$, fietque $\frac{1}{p} = \frac{1}{f} + \frac{npp\cos(\varphi - \vartheta)}{f(1 - qq)uu} + \frac{2nP}{fp}$ et

$$\frac{qq}{pp} = \frac{kk}{ff} - \frac{2np\cos(\varphi - \vartheta)}{f(1 - qq)uu} + \frac{2n}{fw} - \frac{2nQ}{f} + \frac{2nR}{f} + \frac{2nP(1 + qq)}{fpp}$$

et $\qquad \frac{dv\sqrt{(f - 2nP)}}{vv} = \frac{qd\varphi \sin s}{p} \sqrt{\left(f - 2nP + \frac{2np^3\cos(\varphi - \vartheta)}{(1 - qq)(1 + q\cos s)uu}\right)},$

seu $\qquad \frac{dv}{vv} = \frac{qd\varphi \sin s}{p} \sqrt{\left(1 + \frac{2npp\cos(\varphi - \vartheta)}{(1 - qq)(1 + q\cos s)uu}\right)}.$

Vel si nullam approximationem admittamus, erit

$$\frac{1}{p} = \frac{1}{f} + \frac{npp\cos(\varphi - \vartheta)}{f(1 - qq)uu} + \frac{2nP}{fp},$$

$$\frac{qq}{pp} = \frac{kk}{ff} + \frac{2n}{fw} - \frac{2nQ}{f} + \frac{2nR}{f} - \frac{3np\cos(\varphi - \vartheta)}{f(1 - qq)uu} + \frac{npp\cos(\varphi - \vartheta)}{ff(1 - qq)uu} + \frac{2nP}{ffp} + \frac{2nPqq}{fpp},$$

hincque $\qquad \frac{dv\sqrt{(f - 2nP)}}{vv} = \frac{qd\varphi \sin s}{p} \sqrt{\left(f - 2nP + \frac{2np^3\cos(\varphi - \vartheta)}{(1 - qq)(1 + q\cos s)uu}\right)},$ seu

$$\frac{dv}{vv}=\frac{qd\varphi\sin s}{p}\sqrt{\left(1+\frac{2np^3\cos(\varphi-\vartheta)}{(f-2nP)(1-qq)(1+q\cos s)uu}\right)}.$$

Est autem
$$\frac{dv}{vv}=\frac{dp}{pp}-\frac{(pdq-qdp)}{pp}\cos s+\frac{qds\sin s}{p}.$$

Cum nunc sint p et q proxime constantes, erit

$$\frac{dp}{pp}=\frac{nf(d\varphi-d\vartheta)\sin(\varphi-\vartheta)}{(1-kk)uu}+\frac{2nfdu\cos(\varphi-\vartheta)}{(1-kk)u^3}-\frac{2nuv^3d\varphi}{ff}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\sin(\varphi-\vartheta),$$

$$\frac{2q}{p}\cdot\frac{(pdq-qdp)}{pp}=-\frac{2n}{fu^3}\Big(vdv-udv\cos(\varphi-\vartheta)+uvd\varphi\sin(\varphi-\vartheta)-\frac{(1+kk)uvd\varphi}{(1+k\cos s)^2}\sin(\varphi-\vartheta)\Big)$$

$$-\frac{2n(1+kk)vd\varphi\sin(\varphi-\vartheta)}{f(1+k\cos s)^2uu}+\frac{2nvd\vartheta\sin(\varphi-\vartheta)}{fuu}+\frac{4nvdu\cos(\varphi-\vartheta)}{fu^3}$$

$$+\frac{2n(d\varphi-d\vartheta)\sin(\varphi-\vartheta)}{(1-kk)uu}+\frac{4ndu\cos(\varphi-\vartheta)}{(1-kk)u^3},$$

quibus valoribus substitutis, ob $dv=\frac{kvvd\varphi\sin s}{f}$ proxime, fit

$$\frac{dv}{vv}=\frac{qds\sin s}{p}+\frac{nv^3d\varphi\sin s\cos s}{fw^3}-\frac{nuv^3d\varphi\sin s}{ffw^3}(2\sin s\sin(\varphi-\vartheta)+\cos s\cos(\varphi-\vartheta)+k\cos s\cos(\varphi-\vartheta-s))$$

$$+\frac{nv^3d\varphi\sin^2s\sin(\varphi-\vartheta)}{ff(1-kk)uu}(3+3k\cos s-2kk+kk\cos^2s-k^3\cos s)-\frac{nvd\vartheta\sin^2s\sin(\varphi-\vartheta)}{(1-kk)uu}+\frac{2nvdu\sin^2s\cos(\varphi-\vartheta)}{(1-kk)u^3}.$$

Ex quibus colligimus

$$\frac{q(d\varphi-ds)}{p}=\frac{nv^3d\varphi\cos s}{fw^3}-\frac{nuv^3d\varphi}{ffw^3}(2\sin s\sin(\varphi-\vartheta)+\cos s\cos(\varphi-\vartheta)+k\cos s\cos(\varphi-\vartheta-s))$$

$$-\frac{nkvd\varphi\cos(\varphi-\vartheta)}{(1-kk)uu}+\frac{nv^3d\varphi\sin s\sin(\varphi-\vartheta)}{ff(1-kk)uu}(3+3k\cos s-2kk+kk\cos^2s-k^3\cos s)$$

$$-\frac{nvd\vartheta\sin s\sin(\varphi-\vartheta)}{(1-kk)uu}+\frac{2nvdu\sin s\cos(\varphi-\vartheta)}{(1-kk)u^3},$$

quae formula ita repraesentari potest:

$$\frac{q(d\varphi-ds)}{p}=\frac{nv^3d\varphi\cos s}{fw^3}-nd.\left(\frac{v\sin s\cos(\varphi-\vartheta)}{(1-kk)uu}\right)$$

$$-\frac{nuv^3d\varphi}{ff}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\left(2\sin s\sin(\varphi-\vartheta)+\cos s\cos(\varphi-\vartheta)+k\cos s\cos(\varphi-\vartheta-s)\right)$$

ita ut pro motu lineae absidum sit

$$\varphi-s=\text{Const.}-\frac{nfv\sin s\cos(\varphi-\vartheta)}{k(1-kk)uu}+\frac{n}{k}\int\frac{v^3d\varphi\cos s}{w^3}$$

$$-\frac{n}{fk}\int uv^3d\varphi\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\left(2\sin s\sin(\varphi-\vartheta)+\cos s\cos(\varphi-\vartheta)+k\cos s\cos(\varphi-\vartheta-s)\right)$$

in quibus terminis minimis est $v=\frac{f}{1+k\cos s}$.

Denique quo haec ad tempus revocari queant, erit $vvd\varphi = dt\sqrt{2g(L+M)}\,(f-2nP)$, ita ut proxime $vvd\varphi = dt\sqrt{2fg(L+M)}$ et $ds = d\varphi$.

186. **Coroll. 1.** Si corpus N motu regulari circa L circumferatur, in orbita, cujus semiparameter $= b$, excentricitas $= e$ et anomalia vera $= r$, ut sit $u = \frac{b}{1+e\cos r}$, erit

$$d\vartheta = dr, \qquad uud\vartheta = dt\sqrt{2bg\,(L+N)} \qquad \text{et} \qquad du = \frac{euud\vartheta\sin r}{b}.$$

Unde posito $\sqrt{\frac{b(L+N)}{f(L+M)}} = m$, erit proxime

$$uud\vartheta = mvvd\varphi, \quad \text{seu} \quad d\vartheta = \frac{mvvd\varphi}{uu} = dr \quad \text{et} \quad du = \frac{mevvd\varphi\sin r}{b},$$

ita ut in superioribus formulis fractione n affectis omnia elementa ad $d\varphi$ reducantur.

187. **Coroll. 2.** His differentialibus introductis etiam differentiale dw ad $d\varphi$ perducemus, obtinebimus enim ob $dv = \frac{kvvd\varphi\sin s}{b}$ proxime

$$wdw = \frac{kvvd\varphi\sin s}{f}(v - u\cos(\varphi-\vartheta)) + \frac{mevvd\varphi\sin r}{b}(u - v\cos(\varphi-\vartheta)) + \frac{vd\varphi\sin(\varphi-\vartheta)}{u}(uu - mvv).$$

188. **Coroll. 3.** Ex relatione cognita, quae inter differentialia $d\varphi$, $d\vartheta$, ds, dr, dv et du locum habet, colligi poterunt valores formularum integralium P, Q et R, unde semiparameter variabilis p cum excentricitate q accuratius definientur.

189. **Scholion 1.** Haec solutio per approximationes instituenda isti innititur fundamento, quod termini littera n affecti sint valde parvi; quod duplici modo evenire potest, vel si ipse numerus n fuerit minimus, dum inter quantitates v, u, w non enormis inaequalitas versatur, vel si saltem termini $\frac{n}{w}$ et $\frac{n}{u}$ prae $\frac{1}{v}$ sint quam minimi, quod fieri potest, etiamsi n sit numerus valde magnus. Ita si L sit terra, M luna, et N sol, fractio $\frac{N}{L+M} = n$ quidem est maxima. Verum distantia terrae a sole u tantopere superat distantiam lunae a terra v, ut termini $\frac{n}{w}$ et $\frac{n}{u}$ nihilominus sint perquam exigui prae $\frac{1}{v}$. At si L sit terra, M sol et N luna, ut perturbationes motus solis apparentis a luna ortae investigentur, erit n fractio minima, et distantiae v et w praemagnae respectu distantiae u; interim tamen quantitas $\frac{n}{u}$ prae $\frac{1}{v}$ tanquam evanescens est spectanda, hocque casu terminus $\frac{1}{w}$ prae $\frac{1}{u}$ rejici poterit. Quodsi porro L sit sol, M vero et N duo quicunque planetae primarii, erit n fractio minima, et quia distantiae u, v, w non adeo sunt inaequales, ut una prae reliquis contemni queat, termini $\frac{n}{w}$ et $\frac{n}{u}$ utique prae $\frac{1}{v}$ rejici poterunt.

190. **Scholion 2.** Terminos autem $\frac{n}{w}$ et $\frac{n}{u}$ tam parvos prae $\frac{1}{v}$ esse oportet, ut termini inde nati per nn affecti nullius futuri essent momenti, quemadmodum etiam in solutione hic exposita omnes terminos, qui altiores ipsius n potestates essent complexuri, rejecimus. Sin autem etiam termini per nn affecti attentionem mereantur, in solutione quidem omnia manerent, donec ad differen-

tialia quantitatum p et q eruenda descendimus, quae accuratius usque ad terminos per ... evolvi deberent, hoc autem modo in ambages inextricabiles incideremus. Verum hic labor etiam necessarius videtur, quando termini per nn affecti per se spectati sunt minimi, quoniam ... integrationem interdum termini multo majores nasci possunt; ita si in formula differentiali occu... hujusmodi terminus $nnd\varphi\cos(\alpha+n\varphi)$, is quidem ob factorem nn elidendus videri posset, sed ... integrationem inde emergit terminus $n\sin(\alpha+n\varphi)$ ad eum ordinem pertinens, quem minime negligere volebamus. Ex quo perspicuum est hunc modum approximandi, quatenus hujusmodi termini ordinibus negligendis occurrunt, maxime esse lubricum, propterea quod termini haud levis momenti excludantur. Atque hoc potissimum in motus lunae investigatione observandum est, ubi ob ... causam ejusmodi termini ingrediuntur, quorum valores a terminis quadrato nn affectis vel ... altioribus potestatibus pendent, qui cum nonnisi difficillime per theoriam eruantur, expedit eorum valores ex observationibus definire.

191. Scholion 3. Formulae nostrae pro p et q inventae ideo non parum intricatae pro... quod in membro $\frac{nv\cos(\varphi-\vartheta)}{uu}$ naturam quantitatis v spectavimus, ejusque loco valorem $\frac{p}{1+q\cos s}$ substituimus, quod cum in formulis Q et R non fecerimus, etiamsi et hi ab v pendeant, etiam illi ... jure illi substitutioni supersedere poterimus. Statuamus ergo brevitatis gratia

$$\frac{1}{w}-\frac{v\cos(\varphi-\vartheta)}{uu}-Q+R=S \quad \text{et posito} \quad C=\frac{f}{2} \quad \text{et} \quad D=\frac{kk-1}{2f},$$

sequentes aequationes resolvendae proponuntur

$$vv\,d\varphi=dt\sqrt{2g(L+M)(f-2nP)} \quad \text{et}$$

$$\frac{dv\sqrt{(f-2nP)}}{vv}=d\varphi\sqrt{\left(\frac{kk-1}{f}+\frac{2}{v}+2nS-\frac{f}{vv}+\frac{2nP}{vv}\right)}\ (*).$$

Statuamus nunc $v=\frac{p}{1+q\cos s}$, et haec formula signo radicali implicata fit

$$\frac{kk-1}{f}+\frac{2}{p}+2nS-\frac{f}{pp}+\frac{2nP}{pp}+\frac{2q\cos s}{p}-\frac{2fq\cos s}{pp}+\frac{4nPq\cos s}{pp}-\frac{fqq\cos^2 s}{pp}+\frac{2nPqq\cos^2 s}{pp}.$$

Evanescant primo termini per $\cos s$ affecti, eritque

$$1-\frac{f}{p}+\frac{2nP}{p}=0, \quad \text{seu} \quad p=f-2nP;$$

hoc modo illa formula abit in

(*) Si excentricitas k evanescat, alio modo calculum tractari oportet; erit enim
$$\frac{1}{v}=\mathfrak{A}+\mathfrak{B}\cos\eta+\mathfrak{C}\cos^2\eta+\text{etc.} \quad \text{et} \quad \frac{dv}{vv}=d\varphi\sqrt{(A+B\cos\eta+C\cos^2\eta+D\cos^3\eta+\text{etc.})}$$
Cum nunc dv factorem obtineat $\sin\eta$, necesse est, ut sit $A\pm B+C\pm D+E$ etc. $=0$, $A+C+E+$ etc. $=0$ et $B+D+$ etc. $=0$. Simili modo poni debet $P=\ldots+\cos\eta+\cos^2\eta\ldots$ et $S=\ldots+\cos\eta+\cos^2\eta\ldots$ Haec methodus aptior videtur illa, qua omnes termini ad sinus vel cosinus angulorum multiplorum ipsius $\eta=\varphi-\vartheta$ reducuntur.

$$\frac{kk-1}{f}+\frac{1}{p}+2nS-\frac{qq}{p}\cos^2 s.$$

ergo $\frac{qq}{p}=\frac{kk-1}{f}+\frac{1}{p}+2nS$, eritque

$$\frac{dv\sqrt{p}}{vv}=\frac{qd\varphi\sin s}{\sqrt{p}},\quad \text{seu}\quad \frac{dv}{vv}=\frac{qd\varphi\sin s}{p}.$$

ergo sit $$p=f-2nP \quad \text{et} \quad qq=1+\frac{(kk-1)p}{f}+2nSp,$$

differentiando $$dp=-2ndP \quad \text{et} \quad 2qdq=\frac{(kk-1)\,dp}{f}+2nSdp+2npdS, \quad \text{hincque}$$

$$\frac{dv}{vv}=\frac{dp}{pp}+\frac{qdp\cos s}{pp}-\frac{(kk-1)\,dp\cos s}{2fpq}-\frac{nSdp\cos s}{pq}-\frac{ndS\cos s}{q}+\frac{qds\sin s}{p},$$

colligitur $$\frac{q\,(d\varphi-ds)\sin s}{p}=\frac{dp}{p}\Big(\frac{1}{v}-\frac{(kk-1)\cos s}{2fq}-\frac{nS\cos s}{q}\Big)-\frac{ndS\cos s}{q},$$

seu $$\frac{q\,(d\varphi-ds)\sin s}{p}=\frac{dp\,(\cos s+2q+qq\cos s)}{2ppq}-\frac{ndS\cos s}{q}.$$

Est vero $$dS=\frac{-vdv+udv\cos(\varphi-\vartheta)-uvd\varphi\sin(\varphi-\vartheta)}{w^3}-\frac{dv\cos(\varphi-\vartheta)}{uu}+\frac{vd\varphi\sin(\varphi-\vartheta)}{uu},$$

$$\text{seu}\quad dS=\frac{-qvv\,d\varphi\sin s}{pw^3}(v-u\cos(\varphi-\vartheta))-\frac{uv\,d\varphi\sin(\varphi-\vartheta)}{w^3}-\frac{qvv\,d\varphi\sin s\cos(\varphi-\vartheta)}{puu}+\frac{vd\varphi\sin(\varphi-\vartheta)}{uu},$$

quo valore substituto orietur

$$\frac{q\,(d\varphi-ds)\sin s}{p}=\frac{nv^3d\varphi\sin s\cos s}{pw^3}-\frac{nuvv\,d\varphi\sin s\cos s\cos(\varphi-\vartheta)}{p}\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big)-\frac{nuvd\varphi\sin^2 s\sin(\varphi-\vartheta)}{(1+q\cos s)^2}\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big)(2+q\cos s),$$

quae divisa per $\frac{q\sin s}{p}$ praebebit

$$d\varphi-ds=\frac{nvv\,d\varphi}{q}\Big(\frac{v\cos s}{w^3}-u\cos s\cos(\varphi-\vartheta)\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big)-\frac{(2+q\cos s)\,u\sin s\sin(\varphi-\vartheta)}{1+q\cos s}\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big)\Big),$$

sive hoc modo

$$d\varphi-ds=\frac{nvv\,d\varphi}{q}\Big(\frac{v\cos s}{w^3}-u\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big)\Big(\cos s\cos(\varphi-\vartheta)-\frac{(2+q\cos s)\sin s\sin(\varphi-\vartheta)}{1+q\cos s}\Big),$$

ubi nullae plane approximationes sunt adhibitae. Tum vero erit

$$P=\int uv^3\,d\varphi\sin(\varphi-\vartheta)\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big) \quad \text{et}$$

$$S=-\int\frac{qv^3\,d\varphi\sin s}{pw^3}+\int\frac{quvv\,d\varphi\sin s\cos(\varphi-\vartheta)}{p}\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big)-\int uvd\varphi\sin(\varphi-\vartheta)\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big),$$

sive $$S=-\int\frac{qv^3\,d\varphi\sin s}{pw^3}+\int\frac{uvv\,d\varphi}{p}\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big)\Big(q\sin s\cos(\varphi-\vartheta)-(1+q\cos s)\sin(\varphi-\vartheta)\Big).$$

etiam $$S=-\int\frac{qv^3\,d\varphi\sin s}{pw^3}-\int\frac{uvv\,d\varphi}{p}\Big(\frac{1}{w^3}-\frac{1}{u^3}\Big)\Big(\sin(\varphi-\vartheta)+q\sin(\varphi-\vartheta-s)\Big).$$

Quibus valoribus integrabilibus definitis habebitur

$$p = f - 2nP, \quad q = \sqrt{\left(\frac{kkp}{f} + 1 - \frac{p}{f} + 2nSp\right)} \quad \text{et}$$

$$dt\sqrt{2g(L+M)(f-2nP)} = vvd\varphi = dt\sqrt{2gp(L+M)},$$

existente $v = \frac{p}{1+q\cos s}$, $dv = \frac{qvvd\varphi \sin s}{p}$ et $w = \sqrt{(vv - 2vu\cos(\varphi - \vartheta) + uu)}$.

Atque haec solutio praecedenti longe praeferenda videtur, cum quod nullis adhuc approximationibus sit restricta, tum vero quod ejus forma simplicior reperiatur.

192. **Problema.** (Fig. 187.) Si corpus N circa corpus L, quod in quiete spectamus, motu regulari feratur, tum vero corpus M non in eodem plano circa L ita moveatur, ut ejus motus ab actione corporis N perturbetur, definire has perturbationes.

Solutio. Ex corpore M, in planum orbitae a corpore N descriptae demittatur perpendiculum MP, et ex P ad rectam fixam LA agatur normalis PQ, vocenturque coordinatae pro corpore M $LQ = x$, $QP = y$ et $PM = z$, sitque distantia $LM = v = \sqrt{(xx + yy + zz)}$. Tum vero pro corpore N sint coordinatae $LR = \mathfrak{x}$, $RN = \mathfrak{y}$ et distantia $LN = u$. Posito ergo angulo $ALN = \vartheta$, erit $\mathfrak{x} = u\cos\vartheta$ et $\mathfrak{y} = u\sin\vartheta$. Deinde ponatur distantia $MN = w$, ut sit $w = \sqrt{((\mathfrak{x}-x)^2 + (\mathfrak{y}-y)^2 + zz)}$. Jam secundum directiones ternarum coordinatarum vires corpus M sollicitantes resolvantur, et cum primo M ad L trahatur vi $= \frac{M(L+M)}{vv}$, hinc nascitur vis

$$\text{sec. } LQ = \frac{-M(L+M)x}{v^3}, \quad \text{sec. } QP = \frac{-M(L+M)y}{v^3}, \quad \text{sec. } PM = \frac{-M(L+M)z}{v^3}.$$

Deinde ad corpus N urgetur vi $= \frac{MN}{ww}$, unde nascitur vis

$$\text{sec. } LQ = \frac{MN(\mathfrak{x}-x)}{w^3}, \quad \text{sec. } QP = \frac{MN(\mathfrak{y}-y)}{w^3}, \quad \text{sec. } PM = \frac{-MNz}{w^3}.$$

Denique cum corpus L ad N sollicitetur vi $= \frac{LN}{uu}$, hac rite in M translata prodit vis

$$\text{sec. } LQ = \frac{-MN\mathfrak{x}}{u^3} \quad \text{et} \quad \text{sec. } QP = \frac{-MN\mathfrak{y}}{u^3}.$$

Ex his viribus formulae motum continentes ita se habebunt

$$ddx = -2gdt^2\left(\frac{(L+M)x}{v^3} - \frac{N(\mathfrak{x}-x)}{w^3} + \frac{N\mathfrak{x}}{u^3}\right),$$

$$ddy = -2gdt^2\left(\frac{(L+M)y}{v^3} - \frac{N(\mathfrak{y}-y)}{w^3} + \frac{N\mathfrak{y}}{u^3}\right),$$

$$ddz = -2gdt^2\left(\frac{(L+M)z}{v^3} + \frac{Nz}{w^3}\right).$$

Ponamus brevitatis gratia $\frac{N}{L+M} = n$, ut habeamus

$$ddx = -2g(L+M)\,dt^2\left(\frac{x}{v^3} + \frac{nx}{w^3} - n\mathfrak{x}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right),$$

$$ddy = -2g(L+M)\,dt^2\left(\frac{y}{v^3} + \frac{ny}{w^3} - n\mathfrak{y}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\right),$$

$$ddz = -2g(L+M)\,dt^2\left(\frac{z}{v^3} + \frac{nz}{w^3}\right).$$

His cum solutione problematis § 169 comparatis, quod ibi erat L hic nobis est $L+M$, ac praeterea

$$X = \frac{nx}{w^3} - nu\cos\vartheta\left(\frac{1}{w^3} - \frac{1}{u^3}\right), \quad Y = \frac{ny}{w^3} - nu\sin\vartheta\left(\frac{1}{w^3} - \frac{1}{u^3}\right), \quad Z = \frac{nz}{w^3}.$$

Sit nunc, solutionem secundum praecepta ibi data prosequendo, recta $L☊$ linea nodorum et ☊ nodus ascendens, ponaturque angulus $AL☊ = \psi$ et inclinatio praesens orbitae a corpore M descriptae ad planum orbitae $N = \omega$; tum vocetur angulus $☊LM = \sigma$, eritque

$$x = v(\cos\sigma\cos\psi - \sin\sigma\sin\psi\cos\omega), \quad y = v(\cos\sigma\sin\psi + \sin\sigma\cos\psi\cos\omega) \quad \text{et} \quad z = v\sin\sigma\sin\omega,$$

erit $d\omega = \frac{d\psi\cos\sigma\sin\omega}{\sin\sigma}$, atque fiat $d\sigma + d\psi\cos\omega = d\varphi$, ut sit φ longitudo corporis M in sua orbita. Quibus positis erit

$$dv^2 + vv\,d\varphi^2 = 2g(L+M)\,dt^2\left(\frac{kk-1}{f} + \frac{2}{v} - 2\int(Xdx + Ydy + Zdz)\right)$$

$$\text{et} \qquad v^4\,d\varphi^2\cos^2\omega = 4g(L+M)\,dt^2\int vv\,d\varphi\cos\omega\,(Xy - Yx)$$

atque
$$d\psi = \frac{2g(L+M)\,dt^2\sin\sigma}{v\,d\varphi}(Y\cos\psi + X\sin\psi - Z\cot\omega).$$

Cum autem sit

$$xdy - ydx = vv\,d\varphi\cos\omega, \quad xdz - zdx = vv\,d\varphi\cos\psi\sin\omega, \quad ydz - zdy = vv\,d\varphi\sin\psi\sin\omega,$$

erit
$$dx = \frac{xdz}{z} - \frac{vv\,d\varphi\cos\psi\sin\omega}{z}, \qquad dy = \frac{ydz}{z} - \frac{vv\,d\varphi\sin\psi\sin\omega}{z}$$

$$\text{et} \qquad \frac{dz}{z} = \frac{dv}{v} + \frac{d\sigma\cos\sigma}{\sin\sigma} + \frac{d\psi\cos\sigma\cos\omega}{\sin\sigma} = \frac{dv}{v} + \frac{d\varphi\cos\sigma}{\sin\sigma}.$$

Pro reductione formularum datarum habemus primo

$$Xy - Yx = nu(x\sin\vartheta - y\cos\vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right), \quad \text{seu}$$

$$Xy - Yx = nuv(\cos\sigma\sin(\vartheta - \psi) - \sin\sigma\cos\omega\cos(\vartheta - \psi))\left(\frac{1}{w^3} - \frac{1}{u^3}\right).$$

Deinde est

$$Xdx + Ydy + Zdz = \frac{nv\,dv}{w^3} - nu\,dv\left(\frac{1}{w^3} - \frac{1}{u^3}\right)(\cos\sigma\cos(\psi - \vartheta) - \sin\sigma\cos\omega\sin(\psi - \vartheta))$$

$$+ nuv\,d\varphi\left(\frac{1}{w^3} - \frac{1}{u^3}\right)(\sin\sigma\cos(\psi - \vartheta) + \cos\sigma\cos\omega\sin(\psi - \vartheta)),$$

atque

$$v^4\, d\varphi^2 \sin^2\omega = -4ng\,(L+M)\, dt^2 \int uv^3\, d\varphi \sin\sigma \sin^2\omega \cos(\psi-\vartheta)\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

$$v^4\, d\varphi^2 \cos^2\omega = -4ng\,(L+M)\, dt^2 \int uv^3\, d\varphi \cos\omega\,(\cos\sigma \sin(\psi-\vartheta) + \sin\sigma \cos\omega \cos(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

unde colligendo fit

$$v^4\, d\varphi^2 = -4ng\,(L+M)\, dt^2 \int uv^3\, d\varphi\,(\sin\sigma \cos(\psi-\vartheta) + \cos\sigma \cos\omega \sin(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$

Ponamus jam brevitatis gratia

$$\int uv^3\, d\varphi\,(\sin\sigma \cos(\psi-\vartheta) + \cos\sigma \cos\omega \sin(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right) = P,$$

$$\int \frac{udv}{w^3} - \int udv\,(\cos\sigma \cos(\psi-\vartheta) - \sin\sigma \cos\omega \sin(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$$
$$+ \int uv\, d\varphi\,(\sin\sigma \cos(\psi-\vartheta) + \cos\sigma \cos\omega \sin(\psi-\vartheta))\left(\frac{1}{w^3}-\frac{1}{u^3}\right) = Q,$$

eritque
$$v^4\, d\varphi^2 = 2g\,(L+M)\, dt^2\,(f-2nP)$$

et
$$dv^2 + v^2\, d\varphi^2 = 2g\,(L+M)\, dt^2\left(\frac{kk-1}{f} + \frac{2}{v} - 2nQ\right),$$

unde fit
$$dv^2(f-2nP) = v^4\, d\varphi^2\left(\frac{kk-1}{f} + \frac{2}{v} - 2nQ - \frac{f}{vv} + \frac{2nP}{vv}\right)$$

et
$$\frac{dv}{vv}\sqrt{(f-2nP)} = d\varphi\sqrt{\left(\frac{kk-1}{f} + \frac{2}{v} - 2nQ - \frac{f}{vv} + \frac{2nP}{vv}\right)}.$$

Quare si ut supra ponamus $v = \frac{p}{1+q\cos s}$, obtinebimus

$$p = f - 2nP, \qquad qq = 1 + \frac{(kk-1)\,p}{f} - 2nQp \qquad \text{et} \qquad \frac{dv}{vv} = \frac{qd\varphi \sin s}{p},$$

ac porro
$$\frac{q\,(d\varphi - ds)\sin s}{p} = \frac{dp\,(\cos s + 2q + qq\cos s)}{2ppq} + \frac{ndQ\cos s}{q}.$$

Postea vero reperimus

$$d\psi = \frac{2ng\,(L+M)\, udt^2 \sin\sigma \sin(\psi-\vartheta)}{vd\varphi}\left(\frac{1}{w^3}-\frac{1}{u^3}\right);$$

et ob $2g\,(L+M)\, dt^2 = \frac{v^4\, d\varphi^2}{p}$ erit

$$d\psi = \frac{nuv^3\, d\varphi \sin\sigma \sin(\psi-\vartheta)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right) \quad \text{et} \quad \frac{d\omega}{\sin\omega} = \frac{nuv^3\, d\varphi \cos\sigma \sin(\psi-\vartheta)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right).$$

Praeterea ex his valoribus nanciscimur

$$w = \sqrt{\left(vv + uu - 2uv\,(\cos\sigma \cos(\psi-\vartheta) - \sin\sigma \cos\omega \sin(\psi-\vartheta))\right)}.$$

Ponamus nunc brevitatis gratia

$$\cos\sigma \cos(\psi-\vartheta) - \sin\sigma \cos\omega \sin(\psi-\vartheta) = \cos\lambda,$$
$$\sin\sigma \cos(\psi-\vartheta) + \cos\sigma \cos\omega \sin(\psi-\vartheta) = \sin\mu,$$

$$P = \int uv^3\,d\varphi \sin\mu\left(\frac{1}{w^3}-\frac{1}{u^3}\right), \qquad Q = \int \frac{v\,dv}{w^3} + \int (uu\,d\varphi \sin\mu - u\,dv \cos\lambda)\left(\frac{1}{w^3}-\frac{1}{u^3}\right).$$

Cum jam sit $dp = -2nuv^3\,d\varphi \sin\mu\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$, erit

$$\frac{g\,(d\varphi - ds)\sin s}{p} = -\frac{nuv^3\,d\varphi \sin\mu\,(\cos s + 2q + qq\cos s)}{ppq}\left(\frac{1}{w^3}-\frac{1}{u^3}\right) + \frac{nv\,dv\cos s}{qw^3}$$

$$+\frac{nuv\,d\varphi \sin\mu\cos s}{q}\left(\frac{1}{w^3}-\frac{1}{u^3}\right) \qquad -\frac{nu\,dv\cos\lambda\cos s}{q}\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

seu

$$\frac{g\,(d\varphi - ds)}{p} = -\frac{nuv^3\,d\varphi\sin s\sin\mu}{pp}(2+q\cos s)\left(\frac{1}{w^3}-\frac{1}{u^3}\right) + \frac{nv^3\,d\varphi\cos s}{pw^3} - \frac{nuvv\,d\varphi\cos s\cos\lambda}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

posito $w = \sqrt{(vv + uu - 2uv\cos\lambda)}$, unde patet λ denotare angulum MLN. Cum ergo sit $d\eta = d\varphi - d\psi\cos\omega$, erit

$$d\sigma = d\varphi - \frac{nuv^3\,d\varphi\sin\sigma\cos\omega\sin(\psi-\vartheta)}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right) \qquad \text{et}$$

$$ds = d\varphi - \frac{nv^3\,d\varphi\cos s}{qw^3} + \frac{nuv^3\,d\varphi\sin s\sin\mu}{pq}(2+q\cos s)\left(\frac{1}{w^3}-\frac{1}{u^3}\right) + \frac{nuvv\,d\varphi\cos s\cos\lambda}{q}\left(\frac{1}{w^3}-\frac{1}{u^3}\right);$$

tum vero ob $dv = \frac{qvv\,d\varphi\sin s}{p}$ fit

$$dQ = \frac{qv^3\,d\varphi\sin s}{pw^3} + uv\,d\varphi\left(\sin\mu - \frac{qv\cos\lambda\sin s}{p}\right)\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

unde per integrationem valor ipsius Q colligi debet. Denique pro ratione temporis habemus

$$dt\sqrt{2g\,(L+M)} = \frac{vv\,d\varphi}{\sqrt{p}}.$$

Quodsi jam motus corporis N sit regularis ponaturque $u = \frac{b}{1+e\cos r}$, erit

$$dt\sqrt{2g\,(L+N)} = \frac{uu\,d\vartheta}{\sqrt{b}} \quad \text{et} \quad du = \frac{euu\,d\vartheta\sin r}{b};$$

quare posito $\sqrt{\frac{L+M}{L+N}} = \frac{1}{m}$, fit $\frac{1}{m} = \frac{vv\,d\varphi\sqrt{b}}{uu\,d\vartheta\sqrt{p}}$, hinc $d\vartheta = \frac{mvv\,d\varphi\sqrt{b}}{uu\sqrt{p}}$ et

$$du = \frac{mevv\,d\varphi\sin r\sqrt{b}}{b\sqrt{p}} = \frac{mevv\,d\varphi\sin r}{\sqrt{bp}} \quad \text{et} \quad dr = d\vartheta.$$

193. **Coroll. 1.** Cum termini littera n affecti sint minimi, primo his terminis penitus neglectis habebimus $p = f$, $q = k$, $ds = d\varphi$, $v = \frac{f}{1+k\cos s}$, $d\sigma = d\varphi$ et $d\psi = 0$, $d\omega = 0$, quibus valoribus corporis M motus regularis inducitur.

194. **Coroll. 2.** Deinde hi ipsi valores in terminis littera n affectis adhibeantur, ex quibus per integrationem primo quantitates P et Q, tum vero anguli s, σ, ψ et ω investigentur, quibus inventis erit accuratius $p = f - 2nP$ et $q = \sqrt{\left(\frac{kkp}{f} + \frac{2nP}{f} - 2nQp\right)}$, hincque $v = \frac{p}{1+q\cos s}$.

195. **Coroll. 3.** Porro hi valores correcti in formulas integrales introducantur, ac denuo quantitates P et Q quam anguli s, σ, ψ et ω quaerantur, qui valores cum vero sint pro... etiam quantitates p, q et v indeque et w accuratius cognoscentur, unde similis operatio ad ma... consensum cum veritate obtinendum suscipi poterit.

196. **Scholion 1.** Hinc intelligitur istum calculum ob formularum complicationem non sol... esse operosissimum, sed etiam alia via singulas harum formularum partes integrandi non patet, ... ut eae in simplices sinus vel cosinus evolvantur, et integrationes omnes ad hujusmodi term... $\int d\varphi \cos \xi$ perducantur, ubi relatio inter $d\varphi$ et $d\xi$ proxime saltem detur. Quodsi enim ... $d\xi = d\varphi(\alpha + \beta \cos x + \text{etc.})$, ubi terminus α sequentes plurimum superet, ob

$$d\varphi = \frac{d\xi}{\alpha} - \frac{\beta\, d\varphi \cos x}{\alpha} - \text{etc.}, \quad \text{fit} \quad \int d\varphi \cos \xi = \frac{1}{\alpha} \sin \xi - \frac{\beta}{\alpha} \int d\varphi \cos x \cos \xi \text{ etc.},$$

$$\text{at} \quad \int d\varphi \cos x \cos \xi = \tfrac{1}{2} \int d\varphi \cos(\xi - x) + \tfrac{1}{2} \int d\varphi \cos(\xi + x),$$

ita ut hic similis ratio integrationis sit adhibenda. Verum si eveniat, ut ipse numerus α sit perquam exiguus, hoc modo parum proficimus, hocque casu si fuerit $x = \mathfrak{b}\xi + \mathfrak{B}$, integrari oporteret hujus modi formulam

$$\frac{d\xi \cos \xi}{\alpha + \beta \cos(\mathfrak{b}\xi + \mathfrak{B}) + \gamma \cos(\mathfrak{c}\xi + \mathfrak{C}) \text{ etc.}},$$

in qua coëfficientes β et γ prae α non sint exigui, sed potius valde magni. Quare si hujusmodi casus occurrant, ista consueta integrandi methodus minime ad scopum est accommodata. Praeterea quantitas irrationalis $w = \sqrt{(vv + uu - 2vu \cos \lambda)}$ maximum affert obstaculum, nisi insignis inaequalitas inter distantias v et u adsit, ita ut fractio $\frac{1}{w^3}$ facile in seriem valde convergentem transmutari possit. Ob has tantas difficultates optandum esset, ut geometrae potius in alias methodos integrandi quae non ad evolutionem in simplices sinus cosinusve adstringerentur, inquirerent, quod negotium si minus successerit, cognitio motuum coelestium non tam ob defectum Mechanicae, quam ob sufficientem Analyseos promotionem arceri est censenda.

197. **Scholion 2.** Quando autem resolutio formulae irrationalis w in seriem convergentem minus commode succedit, quemadmodum imprimis usu venit, quando perturbatio motus cujusdam planetae ab actione alius planetae vel etiam cometae oriunda definiri debet, ob calculi defectum ... alia via relinquitur, nisi ut pro singulis temporis momentis perturbationes ex formulis differentialibus definiantur, ac deinceps in unam summam colligantur. Planeta scilicet vel cometa assumitur, nisi alter planeta adesset, sectionem conicam circa solem secundum regulas Keplerianas esse descriptum, tum vero quasi singulis temporis momentis vis perturbans accedere concipitur, ubi quanta mutatio tam in ipsa orbita, quam in motu inde efficiatur, determinari oportet; id quod, quia tempus minimum accipiatur, ipsae formulae differentiales ostendent. Quodsi deinceps has perturbationes momentaneas in unam summam colligamus, evidens est conclusionem eo fore certiorem, quo minores fuerint temporis particulae, quamquam etiam hinc errores accumulari sunt censendi.

Caput VII.

De perturbatione motus momentanea a vi quacunque sollicitante oriunda.

98. **Problema.** (Fig. 188.) Si corpus, dum circa aliud corpus motu regulari sectionem conicam esset descripturum, per exiguum temporis intervallum a corpore quodam tertio in orbitae suae plano sito sollicitetur, determinare motus perturbationem momentaneam.

Solutio. Mente primum removeamus corpus perturbans et consideremus motum corporis M, qualis spectaretur ex corpore L, dum haec duo corpora L et M sola existerent ac se mutuo attrahent in ratione reciproca duplicata distantiarum. Describet ergo corpus M sectionem conicam BM, cujus alter focus erit in L, sitque B punctum orbitae ab L minime distans, seu absis ima, cujus longitudo a directione fixa LA computata, sit angulus $ALB = \alpha$. Orbitae vero vocetur semiparameter $= p$ et excentricitas $= q$, erit absidis imae distantia $LB = \frac{p}{1+q}$; absidis vero summae distantia ab $L = \frac{p}{1-q}$, unde fit axis transversus $= \frac{2p}{1-qq}$, cujus semissis $\frac{p}{1-qq}$ ponatur $= r$. Versetur jam corpus, cujus motum investigamus, in M, sitque angulus $BLM = s$, qui ejus anomalia vera appellatur, et distantia $LM = v$, erit $v = \frac{p}{1+q\cos s}$; ipsa vero longitudo a directione fixa LA computata sit angulus $ALM = \varphi$, erit utique $\varphi = \alpha + s$ et $\varphi - s = \alpha$. Quodsi jam tempusculo dt corpus ab M in m progredi sumamus, et litterae L et M massas corporum denotent, erit

$$vv\,ds = dt\sqrt{2gp\,(L+M)}, \qquad \text{ideoque} \qquad dt\sqrt{2gp\,(L+M)} = \frac{pp\,ds}{(1+q\cos s)^2},$$

ita ut sit angulus elementaris tempusculo dt confectus

$$MLm = d\varphi = ds = \frac{dt}{vv}\sqrt{2gp\,(L+M)},$$

ubi quidem litterae L et M massas ita denotare sunt intelligendae, ut $\frac{L}{vv}$ exprimat vim absolutam, qua corpora in distantia $= v$ ad L attrahuntur, posita gravitate absoluta $= 1$ in superficie terrae, illic grave uno minuto secundo per altitudinem $= g$ delabi assumitur, ut tempus t in minutis secundis exprimatur. At quantitates L et M etiam ex tempore periodico colligere licet. Cum enim quantitates p et q sint constantes, erit

$$\int\frac{ds}{(1+q\cos s)^2} = \frac{1}{(1-qq)^{\frac{3}{2}}}\,\text{Arc.}\cos\frac{q+\cos s}{1+q\cos s} - \frac{q\sin s}{(1-qq)(1+q\cos s)},$$

erit integrando:

$$t\sqrt{2gp\,(L+M)} = \frac{pp}{(1-qq)^{\frac{3}{2}}}\,\text{Arc.}\cos\frac{q+\cos s}{1+q\cos s} - \frac{ppq\sin s}{(1-qq)(1+q\cos s)},$$

seu ob $\frac{p}{1-qq} = r$ habebitur

$$t\sqrt{2g\,(L+M)} = r\sqrt{r}\cdot\text{Arc.}\cos\frac{q+\cos s}{1+q\cos s} - qr\sqrt{p}\cdot\frac{\sin s}{1+q\cos s},$$

ubi t denotat tempus, quo corpus M ab abside ima B anomaliam veram $BLM = s$ absolvit. Quod si totum tempus periodicum vocetur $= \Theta$ min. sec. posito $s = 360^0 = 2\pi$, obtinebitur

$$\Theta \sqrt{2g(L+M)} = 2\pi r \sqrt{r}, \quad \text{ita ut sit} \quad \sqrt{2g(L+M)} = \frac{2\pi r \sqrt{r}}{\Theta}.$$

His definitis ponamus dum corpus in M versatur, unde motu assignato ulterius esset progressurum, quasi subito in N existere corpus in plano orbitae cujus massa $= N$, voceturque distantia $LN = u$, angulus $ALN = \vartheta$, sitque distantia $MN = \sqrt{(uu - 2uv\cos(\varphi - \vartheta) + vv)} = w$ brevitatis gratia. Ob actionem hujus corporis N, cujus effectum tantum pro tempusculo dt hic definire statuimus, corpus M tempusculo dt non in m sed in μ perveniet, ejusque motus ita perturbabitur, ut, si corpus N, elapso tempusculo dt subito iterum tolleretur, aliam deinceps orbitam esset descripturum, a priori infinite parum recedentem, puta $\beta\mu$, pro qua statuamus longitudinem absidis imae $AL\beta = \alpha + d\alpha$, semiparametrum $= p + dp$, excentricitatem $= q + dq$, et semiaxem transversum $= r + dr$. Nunc autem elapso tempusculo dt erit anomalia vera $= \beta L\mu$, quas mutationes momentaneas ex problemate § 185 ac praecipue ejus scholio § 191 colligamus. Ponamus ergo, uti ibi, brevitatis gratia $\frac{N}{L+M} = n$, tum vero

$$dP = uv^3 d\varphi \sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right) \quad \text{et}$$

$$dS = -\frac{qv^3 d\varphi \sin s}{pw^3} + \frac{q}{p} uvv\, d\varphi \sin s \cos(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right) - uv\, d\varphi \sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right)$$

atque § 191 invenimus fore, posito $v = \frac{p}{1 + q\cos s}$,

$$\text{I.} \quad vv\, d\varphi = dt\sqrt{2g(L+M)}\,(f - 2nP), \qquad \text{II.} \quad dv = \frac{qvv\, d\varphi \sin s}{p},$$

$$\text{III.} \quad p = f - 2nP, \qquad \text{IV.} \quad \frac{qq}{p} = \frac{kk-1}{f} + \frac{1}{p} + 2nS,$$

$$\text{V.} \quad d\varphi - ds = \frac{nvv\, d\varphi}{q}\left(\frac{v\cos s}{w^3} - u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\cos s\cos(\varphi - \vartheta) + \frac{(2 + q\cos s)\sin s \sin(\varphi - \vartheta)}{1 + q\cos s}\right)\right)$$

ubi f denotat semiparametrum et k excentricitatem pro initio temporis t. Quoniam igitur hic istud initium in principio tempusculi dt constituimus, erit nobis $f = p$ et $k = q$, litterae autem p et q denotant earundem valores jam variatos $p + dp$ et $q + dq$, at $d\varphi$ angulum $ML\mu$. Ex quo colligimus

$$dp = -2n dP, \quad \text{et} \quad d\cdot\frac{1-qq}{p} = -2n dS = d\cdot\frac{1}{r}, \quad \text{atque}$$

$$vv\, d\varphi = dt\sqrt{2g(L+M)}\,(p + dp), \quad \text{seu} \quad = dt\left(\sqrt{p} + \frac{dp}{2\sqrt{p}}\right)\sqrt{2g(L+M)}.$$

Variationes ergo tempusculo dt productae ita se habebunt:

1. semiparameter p augmentum capit dp, ut sit

$$dp = -2nuv^3 d\varphi \sin(\varphi - \vartheta)\left(\frac{1}{w^3} - \frac{1}{u^3}\right);$$

2. semiaxis transversus r, ob $\frac{dr}{rr} = 2ndS$, augmentum capit dr, ut sit

$$dr = \frac{-2nqrr\varrho^3 d\varphi \sin s}{pw^3} + 2nrru\varrho d\varphi \left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\frac{q\varrho}{p}\sin s \cos(\varphi - \vartheta) - \sin(\varphi - \vartheta)\right);$$

3. pro variatione excentricitatis q habemus

$$\frac{-2q\,dq}{p} - \frac{(1-qq)\,dp}{pp} = -2ndS, \quad \text{seu} \quad \frac{2q\,dq}{p} = 2ndS + \frac{2n(1-qq)\,dp}{pp},$$

unde fit

$$dq = \frac{-n\varrho^3 d\varphi \sin s}{w^3} + \frac{nu\varrho^3 d\varphi}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\Big((1+q\cos s)\sin s\cos(\varphi - \vartheta) - (2\cos s + q + q\cos^2 s)\sin(\varphi - \vartheta)\Big);$$

4. angulus autem elementaris $d\varphi$ tempusculo dt descriptus omissa particula infinite parva, ita definitur

$$d\varphi = \frac{dt}{\varrho\varrho}\sqrt{2gp(L+M)},$$

ubi, si tempusculo dt valor notabilis tribuatur, quantitatibus p et ϱ valor medius inter eos, quos initio et fine obtinent, assignari poterit;

5. denique cum sit $\varphi - s = \alpha$, variatio momentanea ipsius α erit

$$d\alpha = \frac{n\varrho\varrho\, d\varphi}{q}\left(\frac{\varrho \cos s}{w^3} - u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\cos s\cos(\varphi - \vartheta) + \frac{(2+q\cos s)\sin s\sin(\varphi - \vartheta)}{1+q\cos s}\right)\right),$$

vel etiam

$$d\alpha = \frac{n\varrho^3 d\varphi}{q}\left(\frac{\cos s}{w^3} - \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\Big((1+q\cos s)\cos s\cos(\varphi - \vartheta) + (2+q\cos s)\sin s\sin(\varphi - \vartheta)\Big)\right).$$

Posset hinc etiam variatio in distantia ϱ facta definiri, sed cum semper sit $\varrho = \frac{p}{1+q\cos s}$, praestat quovis tempore ipsam distantiam ϱ definiri. Omnes ergo perturbationes momentaneae tempusculo dt productae ita determinabuntur:

1. Angulus elementaris interea confectus $d\varphi$ fit

$$d\varphi = \frac{dt}{\varrho\varrho}\sqrt{2gp(L+M)}.$$

2. Semiparameter orbitae p accipiet augmentum dp, ut sit

$$dp = -2nu\varrho^3 d\varphi\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\sin(\varphi - \vartheta).$$

3. Semiaxis transversus orbitae $r = \frac{p}{1-qq}$ accipiet augmentum dr, ut sit

$$dr = \frac{2nrr\varrho\varrho\, d\varphi}{p}\left(\frac{-q\varrho\sin s}{w^3} + u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\Big(q\sin s\cos(\varphi - \vartheta) - (1+q\cos s)\sin(\varphi - \vartheta)\Big)\right),$$

sive $$dr = \frac{-2nrr\varrho\varrho\, d\varphi}{p}\left(\frac{q\varrho\sin s}{w^3} + u\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\Big(\sin(\varphi - \vartheta) + q\sin(\varphi - \vartheta - s)\Big)\right).$$

4. Excentricitas q incrementum dq capiet, ut sit

$$dq = nv^3 d\varphi \left(\frac{-\sin s}{w^3} + \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left((1+q\cos s)\sin s\cos(\varphi-\vartheta) - (2\cos s + q + q\cos^2 s)\sin(\varphi-\vartheta)\right)\right)$$

5. Longitudo absidis α capiet augmentum $d\alpha$, ut sit

$$d\alpha = \frac{nv^3 d\varphi}{q}\left(\frac{\cos s}{w^3} - \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left((1+q\cos s)\cos s\cos(\varphi-\vartheta) + (2+q\cos s)\sin s\sin(\varphi-\vartheta)\right)\right)$$

Ex binis postremis formulis colligur fore

$$dq\cos s + qd\alpha\sin s = -2nuvv d\varphi\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\sin(\varphi-\vartheta) = \frac{dp}{v} \quad \text{et}$$

$$dq\sin s - qd\alpha\cos s = nv^3 d\varphi\left(-\frac{1}{w^3} + \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left((1+q\cos s)\cos(\varphi-\vartheta) - q\sin s\sin(\varphi-\vartheta)\right)\right)$$

quarum illa ex differentiatione aequalitatis $\frac{1}{v} = \frac{1+q\cos s}{p}$ sequitur, ob $\frac{dv}{vv} = \frac{qd\varphi\sin s}{p}$ et $d\varphi - ds = \dots$ fit enim $dq\cos s + qd\alpha\sin s = \frac{dp}{v}$.

199. **Coroll. 1.** Ob actionem ergo corporis N singulis momentis elementa sectionis con... immutantur, ac si id subito annihilaretur, corpus M secundum ea elementa, quae ultimo momen... locum habuerint, moveri perget motu regulari.

200. **Coroll. 2.** Parameter nullam patitur mutationem, si fuerit vel $\sin(\varphi-\vartheta) = 0$, vel $w = u$. Illo casu corpus N cum corporibus L et M in directum est situm, ideoque ex L cum M vel in oppositione vel conjunctione conspicitur; hic vero casus locum habet, ubi fuerit $\cos(\varphi-\vartheta) = \dots$

201. **Coroll. 3.** Si fuerit $\varphi - \vartheta = 0$ et $u > v$, erit $w = u - v$, et perturbationes momentaneae praeter $dp = 0$ inveniuntur:

$$dr = 2nrrvvd\varphi\cdot\frac{q\sin s}{p}\left(\frac{1}{ww} - \frac{1}{uu}\right), \quad dq = nvvd\varphi\sin s\left(\frac{1}{ww} - \frac{1}{uu}\right), \quad d\alpha = \frac{-nvvd\varphi\cos s}{q}\left(\frac{1}{ww} - \frac{1}{uu}\right)$$

202. **Coroll. 4.** Eodem porro casu, quo $\varphi - \vartheta = 0$, si sit $u < v$, ac propterea $w = v - u$, erunt perturbationes momentaneae:

$$dr = -2nrrvvd\varphi\cdot\frac{q\sin s}{p}\left(\frac{1}{ww} + \frac{1}{uu}\right), \quad dq = -nvvd\varphi\sin s\left(\frac{1}{ww} + \frac{1}{uu}\right), \quad d\alpha = \frac{nvvd\varphi\cos s}{q}\left(\frac{1}{ww} + \frac{1}{uu}\right)$$

203. **Coroll. 5.** Sin autem sit $\varphi - \vartheta = 180^0$, erit $\cos(\varphi-\vartheta) = -1$ et $w = v + u$, unde praeter $dp = 0$ reliquae perturbationes erunt

$$dr = 2nrrvvd\varphi\cdot\frac{q\sin s}{p}\left(-\frac{1}{ww} + \frac{1}{uu}\right), \quad dq = nvvd\varphi\sin s\left(-\frac{1}{ww} + \frac{1}{uu}\right), \quad d\alpha = \frac{-nvvd\varphi\cos s}{q}\left(-\frac{1}{ww} + \frac{1}{uu}\right)$$

204. **Coroll. 6.** Casu vero, quo fit $w = u$, ubi etiam $dp = 0$, reliquae perturbationes momentaneae sunt:

$$dr = \frac{-2nqrrv^3d\varphi\sin s}{pu^3}, \quad dq = \frac{-nv^3d\varphi\sin s}{u^3}, \quad d\alpha = \frac{nv^3d\varphi\cos s}{qu^3}.$$

205. **Scholion 1.** Quando ergo motus corporis perturbantis N constat, ut ad singula temporis momenta ejus locus assignari possit, tum ope nostrarum formularum perturbationes singulis momentis productae assignari poterunt. Haec autem temporis momenta, etsi in calculo infinite parva sint assumta, tamen plerumque satis notabilia temporis intervalla, veluti horae, dies, quin etiam hebdomades eorum loco assumi licet, siquidem his intervallis exiguae mutationes oriuntur, vel potius quamdiu mutationes tempori fuerint proxime proportionales. Quatenus enim eae a ratione temporis recedunt, eatenus tempus in minores partes secari oportet. Ita hae formulae commode adhiberi poterunt, si quaestio fuerit, quantum motus cujuspiam planetae principalis ab actione alius planetae vel cometae perturbetur, siquidem utriusque motus in idem fere planum incidat. Ex eodem fonte Celeb. Clairaut perturbationem motus cometae jam appariturì, qui retro annis 1682 et 1607 fuerat observatus, feliciter determinavit, quod negotium etsi summopere laboriosum, eo felicius successit, quod perturbatio tantum, quoad in vicinia planetarum Jovis ac Saturni versabatur cometa, fuerat effecta.

206. **Scholion 2.** Expressiones inventae in alias formas transfundi possunt introducendo angulos trianguli LMN. Si enim ponamus hos angulos $MLN = \varphi - \vartheta = z$, $LMN = y$ et $LNM = x$, quoniam sit $x + y + z = 180^0$, erit $u = \frac{v \sin y}{\sin x}$ et $w = \frac{v \sin z}{\sin x}$, quibus valoribus introductis ob

$$d\varphi = \frac{dt}{vv} \sqrt{2gp\,(L + M)} \qquad \text{et} \qquad v = \frac{p}{1 + q \cos s},$$

reperiuntur variationes tempusculo dt productae:

1. pro variatione semiparametri p,

$$dp = \frac{-2nv\,d\varphi \sin^2 x}{\sin^2 y \sin^2 z}(\sin^3 y - \sin^3 z);$$

2. pro variatione semiaxis transversi r,

$$dr = \frac{-2nrr\,d\varphi \sin^2 x}{p \sin^2 y \sin^2 z}\left((1 + q\cos s)(\sin^3 y - \sin^3 z) + q \sin s\,(\sin^2 y \cos y + \sin^2 z \cos z)\right);$$

vel etiam hoc modo

$$dr = \frac{-2nrr\,d\varphi \sin^2 x}{p \sin^2 y \sin^2 z}\left(\sin^3 y - \sin^3 z + q \sin^2 y \sin(y + s) - q \sin^2 z \sin(z - s)\right);$$

3. pro variatione excentricitatis q,

$$dq = \frac{-n\,d\varphi \sin^2 x}{\sin^2 y \sin^2 z}\left(\sin s\,(\sin^2 y \cos y + \sin^2 z \cos z) + \frac{(2\cos s + q + q\cos^2 s)}{1 + q\cos s}(\sin^3 y - \sin^3 z)\right);$$

4. pro variatione longitudinis absidum α,

$$d\alpha = \frac{n\,d\varphi \sin^2 x}{q \sin^2 y \sin^2 z}\left(\cos s\,(\sin^2 y \cos y + \sin^2 z \cos z) - \frac{\sin s\,(2 + q\cos s)(\sin^3 y - \sin^3 z)}{1 + q \cos s}\right).$$

Cum hae formulae non parum sint complicatae, quovis casu oblato non tam facile dici potest, utrum hae variationes fuerint positivae, an negativae? antequam veros earum valores evolverimus. Interim quia ex istis formulis variationes casu $\varphi - \vartheta = z = 0$ colligere haud licet, priores formae in praxi anteferendae videntur.

207. **Scholion 3.** Effectus corporis N in motu corporis M perturbando est ceteris pa[ribus] maximus, si vel distantia $MN=\varpi$, vel $LN=u$ fuerit minima, hoc est si corpus N vel ad M vel ad L proxime accedat; priori autem casu effectus major erit quam posteriori, quoniam ϖ tantum denominatore nostrarum formularum inest, u vero etiam numeratores afficit. Quodsi igitur L sol, M planeta quidam primarius et N cometa in plano orbitae planetae decurrens, motus quidem planetae maxime turbabitur, quando cometa ad eum proxime accedit; verum etiam dum cometa prope solem praeterit, perturbatio erit eo major, quo vicinior fiat soli et quo major fuerit ejus massa. Ita cometae non solum in perigaeo motum terrae perturbant, sed etiam in perihelio. Ceterum si fieri posset, ut alterutra distantiarum ϖ et u prorsus in nihilum abiret, formulae nostrae omni usu destituerentur, quandoquidem perturbationes fuerint infinitae. Casus hic locum esset habiturus, si corpus N subito alteri corporum L vel M ita jungeretur, ut in unum coalesceret, qui etsi per formulas nostras inexplicabilis videtur, tamen in se est facillimus, propterea quod dum duo tantum aderunt corpora, motus erit regularis, in sectione conica procedens, quanquam haec sectio conica diversa erit ab illa, quae ante accessionem massae N fuerit descripta. Atque hic casus, etsi nonnisi per miraculum locum habere potest, dum massa alterius corporum L vel M augeretur, expendi meretur.

208. **Problema.** Si dum corpora L et M se mutuo attrahentia motu regulari feruntur, alterius vel utriusque massa subito augeatur vel minuatur, definire motum subsecuturum.

Solutio. Hactenus ergo corpus M ex L visum descripserit sectionem conicam BM, cujus semiparameter sit $=p$, excentricitas $=q$ et longitudo absidis $ALB=\alpha$; nunc autem sit corporis M longitudo $ALM=\varphi$ et distantia $LM=v$, erit anomalia vera $BLM=\varphi-\alpha=s$ et $v=\frac{p}{1+q\cos s}$, tum vero expositis horum corporum massis per litteras L et M, tempusculo dt describeretur angulus elementaris $MLm=d\varphi=ds=\frac{dt}{vv}\sqrt{2gp\,(L+M)}$. Jam hoc momento perpendatur corporis M situs ac motus; situs quidem cum distantia $LM=v$, tum angulo $ALM=\varphi$ definitur, ac motus primum directione seu angulo BML, tum vero celeritate ipsa per Mm determinatur. Sit ergo angulus $BML=\eta$ et celeritas in $M=\varepsilon$, ita ut jam hae quatuor quantitates v, φ, η et ε tanquam datae sint spectandae, ex quibus praecedentia motus elementa definiri debent, ac primo quidem dum corporum massae sunt L et M, deinde vero dum massae sunt mutatae, puta L' et M'. Primo igitur habemus

$$\operatorname{tang}\eta=\frac{v\,d\varphi}{dv},\quad \text{sed ob}\quad v=\frac{p}{1+q\cos s}\quad \text{est}\quad dv=\frac{pq\,ds\sin s}{(1+q\cos s)^2},$$

quia ergo est $ds=d\varphi$, erit

$$\operatorname{tang}\eta=\frac{v\,(1+q\cos s)^2}{pq\sin s}=\frac{1+q\cos s}{q\sin s}.$$

Deinde hinc est $Mm=\frac{v\,d\varphi}{\sin\eta}=\frac{v\,d\varphi}{1+q\cos s}\sqrt{(1+2q\cos s+qq)}$, ideoque celeritas

$$\varepsilon=\frac{Mm}{dt}=\frac{Mm}{vv\,d\varphi}\sqrt{2gp\,(L+M)}=\frac{\sqrt{2gp\,(L+M)\,(1+2q\cos s+qq)}}{v\,(1+q\cos s)},\qquad \text{seu}\quad =\frac{\sqrt{2gp\,(L+M)}}{v\sin\eta}$$

unde colligimus $p = \frac{ssvv \sin^2 \eta}{2g(L+M)}$; hincque $1 + q \cos s = \frac{p}{v} = \frac{ssv \sin^2 \eta}{2g(L+M)} = q \sin s \operatorname{tang} \eta$. Quocirca erit

$$q \cos s = \frac{ssv \sin^2 \eta}{2g(L+M)} - 1 \quad \text{et} \quad q \sin s = \frac{ssv \sin \eta \cos \eta}{2g(L+M)},$$

unde pro anomalia vera colligitur $\operatorname{tang} s = \frac{ssv \sin \eta \cos \eta}{ssv \sin^2 \eta - 2g(L+M)}$, hincque ipsa excentricitas

$$q = \frac{\sqrt{(s^4 vv \sin^2 \eta - 4g(L+M)\, ssv \sin^2 \eta + 4gg(L+M)^2)}}{2g(L+M)}.$$

Quodsi nunc massae corporum L et M subito in L' et M' fuerint mutatae, his illarum loco positis hae formulae ostendent elementa orbitae deinceps descriptae, quae elementa sint: 1) semiparameter $= p'$, 2) excentricitas $= q'$ et 3) longitudo absidis imae $= \alpha'$, ita ut posita 4^0 anomalia vera $= s'$, sit $\alpha' = \varphi - s'$. Nunc ergo iterum ex statu praecedente elidantur litterae s et η, scilicet

$$s = \frac{\sqrt{2gp(L+M)(1+2q\cos s + qq)}}{p}, \quad \sin \eta = \frac{1 + q \cos s}{\sqrt{(1+2q\cos s + qq)}}, \quad \cos \eta = \frac{q \sin s}{\sqrt{(1+2q\cos s+qq)}},$$

eritque pro elementis variatis

$$p' = \frac{p(L+M)}{L'+M'}, \quad 1 + q' \cos s' = \frac{p(L+M)}{v(L'+M')}, \quad q' \sin s' = \frac{(L+M)\, q \sin s}{L'+M'}$$

ac $d\varphi = ds' = \frac{dt}{vv}\sqrt{2gp'(L'+M')}$. Nova ergo elementa ita pendent a praecedentibus, ut sit

$$\frac{L+M}{L'+M'} = \frac{p'}{p} = \frac{1+q'\cos s'}{1+q\cos s} = \frac{q' \sin s'}{q \sin s},$$

ideoque quantitates p, $1+q\cos s$ et $q \sin s$ in ratione reciproca massarum immutentur.

209. **Coroll. 1.** Si ergo massae L et M in L' et M' mutentur, dum corpus M in abside ima versatur, ob $s = 0$, erit etiam $s' = 0$, sicque linea absidum nullam patitur mutationem, tum vero erit

$$\frac{1+q'}{1+q} = \frac{L+M}{L'+M'}, \quad \text{ideoque} \quad q' = \frac{L+M}{L'+M'} q + \frac{L+M}{L'+M'} - 1, \quad \text{seu} \quad q' = \frac{p'}{p} q + \frac{p'-p}{p},$$

unde excentricitas vel crescit vel decrescit, semper autem parameter $2p$ in ratione reciproca massarum mutatur.

210. **Coroll. 2.** Si mutatio massarum eveniat, dum corpus M per absidem summam transit, qua p abeat in p', ob $s = 180^0$ et $s' = 180^0$, linea absidum non mutatur, sed excentricitas ita mutatur ut sit

$$\frac{p'}{p} = \frac{1-q'}{1-q}, \quad \text{ideoque} \quad q' = \frac{p'}{p} q + \frac{p-p'}{p}.$$

211. **Coroll. 3.** Si eadem mutatio oriatur dum $s = 90^0$, erit

$$\frac{p'}{p} = 1 + q' \cos s' = \frac{q' \sin s'}{q},$$

unde si $\frac{p'}{p} = \lambda$, habebitur

$q'\sin s' = \lambda q$ et $q'\cos s' = \lambda - 1$, ideoque $q' = \sqrt{(\lambda\lambda qq + (\lambda - 1)^2)}$ et tang s' ...

Si mutatio eveniat dum $s = 270^0$, erit

$q'\sin s' = -\lambda q$ et $q'\cos s' = \lambda - 1$, ideoque $q' = \sqrt{(\lambda\lambda qq + (\lambda - 1)^2)}$ et tang s' ...

212. **Coroll. 4.** Posito ergo $p' = \lambda p$, casu $s = 0$, erit

$$q' = \lambda q + \lambda - 1 \quad \text{et semiaxis transversus} \quad r' = \frac{p}{2(q+1) - \lambda(q+1)^2} = \frac{r(1-q)}{2 - \lambda(1+q)}.$$

Casu $s = 180^0$, ob $q' = \lambda q - \lambda + 1$ fit

$$r' = \frac{p}{2(1-q) - \lambda(1-q)^2} = \frac{r(1+q)}{2 - \lambda(1-q)}.$$

Casu $s = 90^0$, ob $q' = \sqrt{(\lambda\lambda qq + (\lambda - 1)^2)}$ fit

$$r' = \frac{p}{2 - \lambda(1+qq)} = \frac{r(1-qq)}{2 - \lambda(1+qq)}.$$

Casu $s = 270^0$ eadem mutatio in axe transverso oritur.

213. **Coroll. 5.** Si tempus periodicum prius ante mutationem sit Θ, et post mutationem $= \Theta'$, ob $\Theta = \frac{2\pi r\sqrt{r}}{\sqrt{2g(L+M)}}$ et $\Theta' = \frac{2\pi r'\sqrt{r'}}{\sqrt{2g(L'+M')}}$, erit

$$\frac{\Theta'}{\Theta} = \frac{r'\sqrt{r'}}{r\sqrt{r}}\sqrt{\frac{L+M}{L'+M'}} = \frac{r'\sqrt{\lambda r'}}{r\sqrt{r}},$$

unde ex variatione axis transversi variatio in tempore periodico orta definiri potest.

214. **Scholion 1.** Si secundum opinionem, quam Newtonus erat amplexus, massa solis ob lucis emissionem continuo imminueretur, hinc mutatio in motu planetarum facta definiri posset. Foret enim $L + M$ quantitas variabilis, qua posita $= S$, erit

$$d\varphi = \frac{dt}{vv}\sqrt{2gpS} \quad \text{et} \quad \frac{S}{S+dS} = \frac{p+dp}{p} = \frac{1 + q\cos s + d.q\cos s}{1 + q\cos s} = \frac{q\sin s + d.q\sin s}{q\sin s} = 1 - \frac{dS}{S}.$$

In hac autem variatione anomalia vera s eatenus tantum mutari est censenda, quatenus linea absidum mutatur; unde posita longitudine absidis imae $\varphi - s = \alpha$, erit $ds = -d\alpha$. Ne autem haec consideratio moram facessat, praestabit hunc casum ex primis principiis evolvisse. Habemus ergo

$$\text{I.} \quad 2dv\,d\varphi + v\,dd\varphi = 0 \quad \text{et} \quad \text{II.} \quad ddv - v\,d\varphi^2 = \frac{-2gS\,dt^2}{vv},$$

quarum illa dat $vv\,d\varphi = C\,dt$, seu $d\varphi = \frac{C\,dt}{vv}$, unde haec fiet

$$ddv = \frac{CC\,dt^2}{v^3} - \frac{2gS\,dt^2}{vv},$$

ubi S spectari debet tanquam functio temporis t. Quae aequatio quantumvis resolutu difficilis tamen solutio ex formulis superioribus petita ipsi satisfacere deprehenditur. Posito enim

$$v = \frac{p}{1 + q\cos s}, \quad \text{fit primo} \quad p = \frac{bC}{S}, \quad \text{tum vero}$$

$$dq\cos s + qd\alpha\sin s = -\frac{dS}{S}(1 + q\cos s) \quad \text{et} \quad dq\sin s - qd\alpha\cos s = -\frac{dS}{S}q\sin s,$$

unde colligitur

$$d\alpha = \frac{-dS\sin s}{Sq} \quad \text{et} \quad dq = \frac{-dS}{S}(\cos s + q), \quad \text{hincque porro}$$

$$d\varphi = \frac{qdt\sin s}{p}\sqrt{2gbC}.$$

Denique ob $d\varphi = \frac{dt}{vv}\sqrt{2gbC}$ fiet

$$ds = \frac{dt\sqrt{2gbC}}{vv} + \frac{dS\sin s}{Sq},$$

unde saltem variationes momentaneae innotescunt.

215. **Scholion 2.** Solutio hujus problematis suppeditat quoque enodationem quaestionis, qua motus planetae, si forte a quapiam causa ictum acceperit, quem deinceps erit prosecuturus, determinatur. Quemcunque enim motum ante ictum habuerit, si per ictum planetae M imprimatur celeritas $= s$ secundum directionem Mm, ut sit angulus $LMB = \eta$ et distantia $LM = v = \frac{p}{1 + q\cos s}$, post ictum erit semiparameter $p = \frac{ssvv\sin^2\eta}{2g(L+M)}$, excentricitas vero q et anomalia vera s per has aequationes definientur

$$q\cos s = \frac{ssv\sin^2\eta}{2g(L+M)} - 1 \quad \text{et} \quad q\sin s = \frac{ssv\sin\eta\cos\eta}{2g(L+M)},$$

tum vero erit post ictum $d\varphi = ds = \frac{dt}{vv}\sqrt{2gp(L+M)}$, unde sectio conica cum ratione motus innotescit. Verum revertamur ad perturbationem motus planetarum investigandam, quae ab attractione tertii cujusdam corporis efficitur, quando hoc corpus extra planum orbitae est situm. Quanquam autem istud corpus quovis momento tanquam quiescens spectamus, ejus tamen loca successiva in plano quodam per L transeunte sita assumamus, quod planum tanquam fixum consideremus, cujus respectu planum orbitae planetae ob actionem continuo mutetur.

216. **Problema.** (Fig. 189.) Si corpus M, quod ad L attractum motu regulari esset progressurum, a tertio quodam corpore N extra planum motus sito attrahatur, determinare perturbationem motus momentaneam.

Solutio. Referat tabula planum, in quo corpus N perpetuo versetur, in eodem simul perpetuo existente corpore L, cujus respectu motum corporis M definiri oportet. Sit LA recta quaedam fixa, ac nunc quidem elapso tempore $= t$ versetur corpus perturbans in N, posito angulo $ALN = \vartheta$ et distantia $LN = u$; corpus vero, cujus motus quaeritur, sit extra planum ALN in M, unde si corpus N abesset, motu regulari in orbita quadam BM esset progressurum, cujus elementa sequenti modo denotentur. Primo sit $L☊$ intersectio ejus orbitae cum plano ALN, et longitudo nodi ascendentis

$AL\Omega=\psi$, atque inclinatio orbitae ad planum $ALN=\omega$. Deinde ipsius orbitae BM sit semipar... $=p$, excentricitas $=q$ et semiaxis transversus $r=\frac{p}{1-qq}$. Nunc autem sit anomalia vera BLM... eritque distantia $LM=v=\frac{p}{1+q\cos s}$. Ponatur porro angulus $\Omega LM=\sigma$, qui vocatur argum... latitudinis, erit pro abside ima B angulus $\Omega LB=\sigma-s$, ac posita longitudine corporis... orbita propria $=\varphi$, erit, uti supra § 192 vidimus, $d\varphi=d\sigma+d\psi\cos\omega$. Hinc denique quaer... duo anguli λ et μ, ut sit

$$\cos\sigma\cos(\vartheta-\psi)+\sin\sigma\cos\omega\sin(\vartheta-\psi)=\cos\lambda \text{ et } \sin\sigma\cos(\vartheta-\psi)-\cos\sigma\cos\omega\sin(\vartheta-\psi)=\ldots$$

erit $\lambda=$ angulo MLN, unde fiet distantia $MN=\sqrt{(vv+uu-2uv\cos\lambda)}$, quae vocetur $=w$. ... in finem quaeratur angulus ν, ut sit $\operatorname{tang}\nu=\frac{v\sin\lambda}{u-v\cos\lambda}$, eritque $w=\frac{v\sin\lambda}{\sin\nu}$. Quodsi nunc pona... brevitatis gratia

$$\frac{N}{L+M}=n \quad \text{et} \quad uv^3 d\varphi\sin\mu\left(\frac{1}{w^3}-\frac{1}{u^3}\right)=dP,$$

$$\frac{qv^3 d\varphi\sin s}{pw^3}+uv\,d\varphi\left(\sin\mu-\frac{qv\cos\lambda\sin s}{p}\right)\left(\frac{1}{w^3}-\frac{1}{u^3}\right)=dQ,$$

erit primo $d\varphi=\frac{dt}{vv}\sqrt{2gp(L+M)}$, ac perturbationes ab actione corporis N tempusculo dt pro... ductae ex § 192 sequenti modo se habere reperiuntur:

Primo pro variatione semiparametri p est $dp=-2nuv^3 d\varphi\sin\mu\left(\frac{1}{w^3}-\frac{1}{u^3}\right)$.

Deinde pro excentricitatis q variatione ob $\frac{qq-1}{p}=\frac{kk-1}{f}-2nQ$, erit differentiando

$$\frac{2q\,dq}{p}+\frac{(1-qq)\,dp}{pp}=\frac{-2nqv^3 d\varphi\sin s}{pw^3}-2nuv\,d\varphi\left(\sin\mu-\frac{qv\cos\lambda\sin s}{p}\right)\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

unde fit $$dq=\frac{-nv^3 d\varphi\sin s}{w^3}+npuv\,d\varphi\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\left(\frac{\cos\lambda\sin s}{1+q\cos s}-\frac{(2\cos s+q+q\cos^2 s)\sin\mu}{(1+q\cos s)^2}\right),$$

quae reducitur ad hanc formam

$$dq=nv^3 d\varphi\left(\frac{-\sin s}{w^3}+\frac{u}{p}\left(\frac{1}{w^3}-\frac{1}{u^3}\right)\left[(1+q\cos s)\cos\lambda\sin s-(2\cos s+q+q\cos^2 s)\sin\mu\right]\right).$$

Hinc cum sit $\frac{qq-1}{p}=-\frac{1}{r}$, erit $\frac{dr}{rr}=-2n\,dQ$; erit pro variatione semiaxis transversi r

$$dr=\frac{-2nqrrv^3 d\varphi\sin s}{pw^3}-2nrruv\,d\varphi\left(\sin\mu-\frac{q\cos\lambda\sin s}{1+q\cos s}\right)\left(\frac{1}{w^3}-\frac{1}{u^3}\right),$$

seu $$dr=\frac{2nrrvv\,d\varphi}{p}\left(\frac{-qv\sin s}{w^3}+u\left(\frac{1}{w^3}-\frac{1}{u^3}\right)(q\cos\lambda\sin s-(1+q\cos s)\sin\mu)\right).$$

Praeterea consecuti sumus

$$ds = d\varphi - \frac{nv^3\,d\varphi\cos s}{qw^3} + \frac{nuvv\,d\varphi}{q}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)\left(\cos\lambda\cos s + \frac{(2+q\cos s)\sin\mu\sin s}{1+q\cos s}\right),$$

$\varphi - s$ denotet longitudinem absidis imae B in orbita, si ea dicatur $= \alpha$, erit $d\alpha = d\varphi - ds$,

$$d\alpha = \frac{nv^3\,d\varphi}{q}\left(\frac{\cos s}{w^3} - \frac{u}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right)((1+q\cos s)\cos\lambda\cos s + (2+q\cos s)\sin\mu\sin s)\right).$$

pro variatione orbitae respectu plani ALN invenimus primo pro longitudine nodi ☊

$$d\psi = \frac{-nuv^3\,d\varphi\sin\sigma\sin(\vartheta-\psi)}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

pro variatione inclinationis ω

$$d\omega = \frac{d\psi\sin\omega}{\operatorname{tang}\sigma} = \frac{-nuv^3\,d\varphi\cos\sigma\sin\omega\sin(\vartheta-\psi)}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

vero pro variatione anguli ☊$LM = \sigma$ habemus $d\sigma = d\varphi - d\psi\cos\omega$, ac proinde

$$d\sigma = d\varphi + \frac{nuv^3\,d\varphi\sin\sigma\cos\omega\sin(\vartheta-\psi)}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right),$$

$\varphi - \sigma$ designet longitudinem nodi ☊ in orbita, si ea dicatur $= \beta$, erit

$$d\beta = \frac{-nuv^3\,d\varphi\sin\sigma\cos\omega\sin(\vartheta-\psi)}{p}\left(\frac{1}{w^3} - \frac{1}{u^3}\right).$$

vero ob $v = \frac{p}{1+q\cos s}$, erit $dv = \frac{qvv\,d\varphi\sin s}{p}$. Quare cum ex dato tempusculo dt habeatur

$$d\varphi = \frac{dt}{vv}\sqrt{2gp\,(L+M)},$$

omnes perturbationes momentaneae pro tempusculo dt obtinentur. Quod quo facilius ad cal- revocetur, fingamus corpus M circa L in distantia $= c$ circulum describere, in eoque tem- dt angulum $d\zeta$ absolvere, eritque

$$d\zeta = \frac{dt}{c\sqrt{c}}\sqrt{2g\,(L+M)}.$$

cum detur angulus $d\zeta$ ex motu medio erit

$$dt\sqrt{2g\,(L+M)} = cd\zeta\sqrt{c}, \quad \text{ideoque} \quad d\varphi = \frac{cd\zeta\sqrt{cp}}{vv}.$$

217. **Coroll. 1.** Anguli λ et μ ita per trigonometriam sphaericam exhiberi possunt. In super- sphaerica (Fig. 190) centro L descripta sint A, M, N, ☊ puncta, per quae rectae LA, LM, transeant, erit $AN = \vartheta$, A☊ $= \psi$, ☊$N = \vartheta - \psi$, ☊$M = \sigma$ et M☊$N = \omega$, fietque continuato arcu M☊ retro in O, ut OM sit quadrans, si ex O per N itidem ducatur ONR, erit $NR = \mu$.

218. **Coroll. 2.** Ducto arcu MR, quia ad utrumque quadrantem est normalis, resolv… triangulum sphaericum $☊MN$, in quo dantur latera $☊M = \sigma$, $☊N = \vartheta - \psi$ et angulus $M☊N$… inventoque latere MN cum angulo $☊MN$, erit $\lambda = MN$ et $\sin\mu = \sin\lambda\cos ☊MN$.

219. **Coroll. 3.** Loco tempusculi dt spatium non solum aliquot horarum sed etiam… dierum capi potest, nisi positio corporis N ratione ipsius M citissime varietur. Tum ex motu… pro hoc temporis spatio colligatur angulus $d\zeta$, indeque erit $vv\,d\varphi = cd\zeta\sqrt{cp}$, quem valorem… singulis perturbationibus momentaneis substitui oportet.

220. **Scholion.** Ex his principiis perturbationes motus cujusque planetae principalis defini… poterunt, quatenus ab actione alius planetae vel etiam cometae oriuntur; ad planetas autem secun… darios, seu satellites, haec methodus minus commode accommodari potest, quandoquidem assumsi… remoto corpore perturbante, motum futurum esse regularem; hinc itaque perturbationes in motu… lunae, quae forte ab actione cujusdam planetae vel cometae proficiscuntur, determinare nequeunt… Sin autem ipse sol ut corpus perturbans consideretur, sine cujus actione luna motum regula… esset habitura, inaequalitates motus lunae hinc concludere licebit; sed quia actio solis est perennis… collectio perturbationum momentanearum conclusionem nimis lubricam reddit. Maximum autem usum… haec methodus praestabit, si actio cujuspiam cometae in motum planetae principalis, per cujus… viciniam cometa transit, investigari debeat: quoniam enim actio cometae non diutius manet sensibilis… quam dum ejus distantia a planeta fuerit valde parva, omnino superfluum foret, totam actionem… quam cometa per totum suum tempus periodicum exerit, exquirere velle, quem in finem integrali… nostrarum formularum exhiberi opus esset. Sufficiet igitur per breve tempus effectum cometae in… orbita cujuspiam planetae perturbanda cognovisse, id quod ope formularum differentialium haud… difficulter praestabitur. Casus autem, quibus cometae ad planetas tam prope accedunt, ut pertur… bationem notabilem efficere queant, vehementer raro accidunt. Ac si cometa anni 1682 secundum… praedictionem Cel. Clairaut hoc anno 1759 revertatur, phaenomena imprimis singularia in motu… terrae ab ejus actione expectari possent, propterea quod in satis exigua a terra distantia praeteri… batur. Operae ergo pretium erit, ope formularum traditarum in perturbationem motus terrae… que orbitae, ab actione hujus cometae oriundam, inquirere, ut deinceps, quando elementa motus… istius cometae accuratius erunt definita, ad hoc exemplum plenior investigatio suscipi possit.

Digressio,

qua effectus Cometae A. 1759 expectati in motu terrae perturbando investigatur.

I. Primo quidem assumo hunc cometam secundum eadem elementa latum iri, quae pro… apparitione A. 1682 sunt determinata. Etsi enim ob actionem Jovis et Saturni ejus tempus peri… dicum quasi biennio fuit retardatum, ob eandemque rationem ejus reliqua motus elementa haud le… mutationes subiisse probabile, tamen quia de eorum valore praesente nihil certi constat, ante…

... motu ea denuo definire licuerit, elementis superioris revolutionis utar. Posita ergo ... a sole distantia = 100000, statuam pro hoc cometa

1. Distantiam perihelii a sole = 58328
2. Semiparametrum = 116656
3. Nodum ascendentem $1^s\ 21^0\ 16'$
   Nodum descendentem $7^s\ 21^0\ 16'$
4. Distantiam nodi desc. a perih. $71^0\ 36'$
5. Inclinationem ad eclipticam $17^0\ 56'$
6. Longitudinem perihelii $10^s\ 2^0\ 52'$.

Motus autem hujus cometae est retrogradus, et a nodo ascendente ad perihelium, indeque ad nodum descendentem pergit.

II. Qui hunc cometam primum mense Januario hujus anni 1759 viderant, suspicantur eum ... Martii per perihelium suum transiisse, ex quo postquam per nodum descendentem fuerit progressus, ad terram proxime accedet. Nodum descendentem autem attinget circa d. 14 Aprilis, ... post hoc tempus loca cometae colligi conveniet. At ex mea theoria motus cometarum elapsis ... diebus post transitum per perihelium habetur $l\,(t+\frac{1}{3}t^3)=l\delta+8{,}4362521$, unde anomalia vera ... angulus a perihelio confectus definitur, quae si vocetur $=\zeta$, erit distantia ejus a sole $=\frac{58328}{\cos^2\frac{1}{2}\zeta}$.

III. Posito ergo cometam ipso meridie die 14 Martii per perihelium transiisse, die 14 Aprilis ... sequentibus loca cometae ita se habebunt:

Diebus a perihelio	A. 1759	Anomalia vera	ejus semissis	distantia a sole	distantia a nodo descend.
31	April. 14^d	$71^0\ 37'$	$35^0\ 49'$	88705	$0^0\ 1'$
32	15	72 56	36 28	90187	1 20
33	16	74 13	37 7	91731	2 37
34	17	75 28	37 44	93254	3 52
35	18	76 41	38 21	94838	5 5
36	19	77 51	38 55	96349	6 15
37	20	78 58	39 29	97916	7 22
38	21	80 3	40 2	99493	8 29
39	22	81 7	40 34	101070	9 31
40	23	82 8	41 4	102611	10 32
41	24	83 8	41 34	104200	11 32
42	25	84 6	42 3	105782	12 30
43	26	85 2	42 31	107360	13 26
44	27	85 57	42 58	108931	14 21
45	28	86 50	43 25	110550	15 14
46	29	87 42	43 51	112155	16 6
47	30	88 32	44 16	113745	16 56
48	Maji 1	89 21	44 41	115380	17 45
49	2	90 8	45 4	116925	18 32
50	3	90 54	45 27	118517	19 18
51	4	91 39	45 50	120150	20 3
52	5	92 23	46 12	121754	20 47
53	6	93 5	46 34	123401	21 29

IV. Nunc quoque ad singulos hos dies loca terrae ex sole visa ex tabulis colligamus, simul distantias ejus a nodo descendente orbitae cometae, qui cadit in $7^s\,21^0\,16'$ notemus. Pro hoc tempore erat locus perihelii terrae in $3^s\,8^0\,39'$, cujus ergo distantia a nodo descendente est $= 4^s\,12^0$

A. 1759	Distantia terrae a sole	Longitudo terrae	Dist. terrae a nodo desc.
Aprilis 14^d	100400	$6^s\ 23^0\ 13'$	$0^s\ 28^0\ 3'$
15	100420	6 24 11	27 5
16	100450	25 10	26 6
17	100480	26 9	25 7
18	100510	27 7	24 9
19	100540	28 6	23 10
20	100565	29 4	22 12
21	100590	7 0 3	21 13
22	100620	1 1	20 15
23	100650	1 59	19 17
24	100675	2 58	18 18
25	100700	3 56	17 20
26	100725	4 55	16 21
27	100750	5 53	15 23
28	100775	6 51	14 25
29	100800	7 49	13 27
30	100825	8 47	12 29
Maji 1	100850	9 45	11 31
2	100875	10 43	10 33
3	100900	11 41	9 35
4	100925	12 40	8 36
5	100950	13 38	7 38
6	100975	14 36	6 40

V. Pro orbita terrae porro sumitur semiaxis transversus $= 100000$ et excentricitas $= 0{,}0169$ unde fit semiparameter $= 97144$. His elementis constitutis patet circa dies 27 et 28 Aprilis co metam terrae fore proximum. Investigemus ergo perturbationes ab actione cometae oriundas in motu terrae ab 25 Aprilis usque ad 30 ejusdem, et constituamus quina intervalla spatio 24 horarum aequalia, ita tempus dt unum diem, et ex motu terrae medio $d\zeta$ angulum $59'\,8''$ denotet, unde elementum $d\varphi$ definiri debet. Cum autem terra continuo propius ad nodum descendentem progre diatur, dum cometa ab eo recedit, angulus $d\varphi$ negative capiendus est.

VI. Repraesentet ergo (Fig. 190) tabula planum orbitae cometae, in quo sit L sol, A perihelium cometae, a quo per arcum parabolicum AN progrediatur. $BM☊$ vero sit orbita terrae a perihelio B per M ad nodum ☊ progredientis, cujus motus respectu cometae ut retrogradus spectari debet, et portio $BM☊$ supra orbitam cometae versabitur. Erit ergo angulus $BL☊ = 132^0\,27'$ et inclinatio orbitae terrae ad orbitam cometae $\omega = 17^0\,56'$. Quodsi nunc terra haereat in M, cometa vero in N, erit $LM =$ $LN = u$, $MN = w$, $BLM = -s$, $ALN = \vartheta$, $AL☊ = \psi = 71^0\,36'$ et $☊LN = \vartheta - \psi$; $☊LM = \sigma$, atque $r = 100000$, $p = 97144$ et $q = 0{,}0169$. Denique positis massis solis, terrae cometae L, M, N, sit $\frac{N}{L+M} = n$, unde calculi perturbationum pro singulis intervallis diurnis ita se habebunt:

Calculus pro intervallo a 25 ad 26 Aprilis.

Cum sit $p = 97144$, $q = 0{,}0169$, $r = 100000$ et $\varpi = 17^0\,56'$, erit $v = 100700$, $u = 105782$, $LM = \sigma = 17^0\,20'$, $s = -115^0\,17'$, $\vartheta - \psi = 12^0\,30'$. Nunc ob $c = 100000$, ob $d\varphi = \frac{-cd\zeta\sqrt{cp}}{vv}$ $d\zeta = 3548''$ colligitur

$lcp = 9{,}9874160$	$lc\sqrt{cp} = 9{,}9937080$
$l\sqrt{cp} = 4{,}9937080$	$ld\zeta = 3{,}5499836$
$lc = 5{,}0000000$	$13{,}5436916$
$lc\sqrt{cp} = 9{,}9937080$	$lvv = 10{,}0060590$
$lv = 5{,}0030295$	$l - d\varphi = 3{,}5376326$
	$4{,}6855749$
	$l - d\varphi = 8{,}2232075.$

Erit ergo pro terminis, ubi $d\varphi$ angulum denotat, $d\varphi = -3449''$, at pro terminis, ubi in partibus radii exprimi debet, $d\varphi = -0{,}016719$. Pro angulis autem λ et μ calculus ita se habebit:

$l\cos(\vartheta - \psi) = 9{,}9895815$	$l\sin(\vartheta - \psi) = 9{,}3353368$
$l\cos\sigma = 9{,}9798158$	$l\cos\omega = 9{,}9783702$
$l\sin\sigma = 9{,}4741146$	$l\cos\omega.\sin(\vartheta - \psi) = 9{,}3137070$
$9{,}9693973$	$l\sin\sigma = 9{,}4741146$
$9{,}4636961$	$l\cos\sigma = 9{,}9798158$
	$8{,}7878216$
	$9{,}2935228$
$+0{,}93196$	$+0{,}29087$
$+0{,}06135$	$-0{,}19657$
$\cos\lambda = +0{,}99331$	$\sin\mu = 0{,}09430$
$\lambda = 6^0,\,38'$	$\mu = 5^0,\,25'.$

Hinc pro distantia $LN = \omega = \frac{v\sin\lambda}{\sin\nu}$ existente $\operatorname{tang}\nu = \frac{v\sin\lambda}{u - v\cos\lambda}$.

$lv = 5{,}0030295$	$u = 105782$
$l\sin\lambda = 9{,}0626386$	$v\cos\lambda = 100026$
$l\cos\lambda = 9{,}9970829$	$u - v\cos\lambda = 5756$
$lv\sin\lambda = 4{,}0656681$	$lv\sin\lambda = 4{,}0656681$
$lv\cos\lambda = 5{,}0001124$	$l(u - v\cos\lambda) = 3{,}7601208$
$l\sin\nu = 9{,}9524188$	$l\operatorname{tang}\nu = 10{,}3055473$
$l\omega = 4{,}1132493$	$\nu = 63^0,\,40'$
$lc = 5{,}0000000$	$\omega = 12979$
$l\frac{c}{\omega} = 0{,}8867507$	$lu = 5{,}0244118$
$l\frac{c^3}{\omega^3} = 2{,}6602521$	$l\frac{c}{u} = 9{,}9755882$
$\frac{c^3}{\omega^3} = 457{,}354$	$l\frac{c^3}{u^3} = 9{,}9267646$
	$\frac{c^3}{u^3} = 0{,}84482,$

ergo $\frac{c^3}{w^3} - \frac{c^3}{u^3} = 456,509$ et $lc^3\left(\frac{1}{w^3} - \frac{1}{u^3}\right) = 2,6594494.$

Cum nunc sit

$$dp = -\frac{2nuv^3}{c^3} d\varphi \sin\mu\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right)$$

reperietur variatio semiparametri p;

$$\begin{aligned} l\frac{v^3}{c^3} &= 0,0090885 \\ lu &= 5,0244118 \\ l\sin\mu &= 8,9749624 \\ l.A &= 2,6594494 \\ l - d\varphi &= 8,2232075 \\ \hline & \ 4,8911196 \end{aligned}$$

erit ergo

$$dp = 2n \,.\, 77825$$

seu $dp = 155650\, n.$

Unde si massa cometae aequalis esset massae terrae, foret $n = \frac{1}{227000}$, ideoque proxime $dp = \dots$ si autem cometa massam haberet Jovi aequalem, foret $n = \frac{1}{1033}$, ideoque $dp = 151$, qui effectus intervallo unius diei productus satis esset notabilis, cum sit $p = 97144$, ideoque abiret in 97295, sicque parte $\frac{1}{643}$ augeretur.

Pro variatione semiaxis transversi $r = 100000$ habemus hanc formulam:

$$dr = -\frac{2nqrr}{p} \cdot \frac{v^3}{c^3} \cdot \frac{c^3}{w^3} \cdot d\varphi \sin s - \frac{2nrruv}{c^3} d\varphi\left(\sin\mu - \frac{qv}{p}\cos\lambda\sin s\right)\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right)$$

cujus formulae calculus ita se habet:

$$\begin{aligned} lqr^2 &= 8,2278867 \\ lp &= 4,9874160 \\ \hline & \ 3,2404707 \\ l\frac{v^3}{c^3} &= 0,0090885 \\ l\frac{c^3}{w^3} &= 2,6602521 \\ l - d\varphi &= 8,2232075 \\ l - \sin s &= 9,9562678 \\ \hline & \ 4,0892866 \\ \text{pars I} &= -2n.12283 \\ \text{pars II} &= +2n.89456 \\ \hline dr &= +2n.77173 \\ dr &= +154346n \end{aligned}$$

$$\begin{aligned} \sin\mu &= 0,09430 \\ \hline l\frac{q}{p} &= 3,2404707 \\ lv &= 5,0030295 \\ l\cos\lambda &= 9,9970829 \\ l - \sin s &= 9,9562678 \\ \hline & \ 8,1968509 \\ -\frac{qv}{p}\cos\lambda\sin s &= +0,01573 \\ \hline \dots\dots &\ 0,11003 \\ \hline l\dots\dots &\ 9,0415111 \\ lu &= 5,0244118 \\ l\frac{v}{c} &= 0,0030295 \\ l - d\varphi &= 8,2232075 \\ l\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right) &= 2,6594494 \\ \hline & \ 4,9516093. \end{aligned}$$

Semiaxis ergo transversus fere par augmentum accipit atque semiparameter, atque hac actione tempus periodicum augetur in ratione 1 ad $1 + 2{,}31519n$, seu annus augmentum capiet

$$= 845\,n \text{ dierum} = 18280\,n \text{ hor.} = 1096800\,n \text{ min.},$$

unde si cometa terrae esset aequalis, augmentum anni hinc natum foret $= 4', 50''$.

Pro excentricitate q, cum sit $p = (1 - qq)r$, erit

$$qq = 1 - \frac{p}{r} \qquad \text{et} \qquad 2qdq = \frac{-rdp + pdr}{rr} = -\frac{dp}{r} + \frac{pdr}{rr}:$$

Fit ergo hic calculus

$ldp =$	$5{,}1921491$	$lp =$	$4{,}9874160$
$lqr =$	$3{,}2278867$	$ldr =$	$5{,}1884954$
	$1{,}9642624$		$10{,}1759114$
	$-92{,}100\,n$	$lqn =$	$8{,}2278867$
			$1{,}9480247$
			$+88{,}720\,n$

ergo $\qquad dq = -46{,}05\,n + 44{,}36\,n = -1{,}69\,n,$

unde patet excentricitatem fere nullam pati mutationem, nisi massa cometae plurimum superet massam terrae; si sit aequalis massae Jovis, fiet $dq = -0{,}00164$ et $q + dq = 0{,}01426$, unde aequatio centri valde imminueretur.

Pro variatione perihelii in orbita, si ponamus angulum $☊ LB = \alpha$, formula supra inventa ita exprimatur:

$$d\alpha = \frac{nv^3 d\varphi}{qc^3}\left(\frac{c^3 \cos s}{w^3} - \frac{u}{p}\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right)\left((1 + q\cos s)\cos\lambda\cos s + (2 + q\cos s)\sin\mu\sin s\right)\right),$$

quae ergo ita evolvetur ob $1 + q\cos s = \frac{p}{v}$:

$lp =$	$4{,}9874160$	$2 + q\cos s =$	$1{,}96469$
$lv =$	$5{,}0030295$	$l(2 + q\cos s) =$	$0{,}2933161$
$l(1 + q\cos s) =$	$9{,}9843865$	$l\sin\mu =$	$8{,}9749624$
$l\cos\lambda =$	$9{,}9970829$	$l\sin s =$	$-9{,}9562678$
$l\cos s =$	$-9{,}6305243$		$-9{,}2245463$
	$-9{,}6119937$		

$$\text{pars postrema} = -0{,}40925 - 0{,}16770 = -0{,}57695$$

l partis postr. $=$	$-9{,}7611382$	$l\frac{c^3}{w^3} =$	$2{,}6602521$
$l\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right) =$	$+2{,}6594494$	$l\cos s =$	$-9{,}6305243$
$lu =$	$5{,}0244118$		$-2{,}2907764$
	$-7{,}4449994$	l aggr. $=$	$1{,}9612787$
$lp =$	$4{,}9874160$	$l\frac{v^3}{c^3} =$	$0{,}0090885$
	$-2{,}4515834$	$ld\varphi =$	$-3{,}5376326$
pars post. $=$	$+286{,}803$		$-5{,}5079998$
pars prior $=$	$-195{,}333$	$lq =$	$8{,}2278867$
aggreg. $=$	$+91{,}470$		$-7{,}2801131$

Ergo erit $\qquad d\alpha = -19059570\,n$ min. sec.

Cum igitur angulus α minuatur, perihelium in orbita secundum seriem signorum promov... et quidem hoc die, si cometa terrae esset aequalis, per $84''$.

Porro pro variatione nodi ☊ posito angulo $AL☊ = \psi$, erit

$$d\psi = -\frac{nu}{p}\cdot\frac{v^3}{c^3}\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right)d\varphi \sin\sigma \sin(\vartheta-\psi)$$

et pro variatione inclinationis $$d\omega = \frac{d\psi \sin\omega}{\operatorname{tang}\sigma};$$

calculus ergo instituatur ut sequitur:

$l\frac{u}{p} = 0{,}0369958$

$l\frac{v^3}{c^3} = 0{,}0090885$

$l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) = 2{,}6594494$

$ld\varphi = -3{,}5376326$

$l\sin\sigma = 9{,}4741146$

$l\sin(\vartheta-\psi) = 9{,}3353368$

$-5{,}0526177$

ergo $d\psi = +112880\,n$ min. sec.

$ld\psi = 5{,}0526177$

$l\sin\omega = 9{,}4884240$

$4{,}5410417$

$l\operatorname{tang}\sigma = 9{,}4942988$

$ld\omega = 5{,}0467429$

ergo $d\omega = +111364\,n$ min. sec.

unde linea nodorum $L☊$ in orbita cometae promovetur angulo $d\psi = 112880\,n$ min. sec. et inclinatio orbitae terrestris augetur angulo $d\omega = 111364\,n$ min. sec., quae mutationes circiter 170 vicibus sunt minores ea, quam linea absidum terrae experitur.

Calculus pro intervallo a 26 ad 27 Aprilis.

Cum sit $p = 97144$; $q = 0{,}0169$; $r = 100000$, et $\omega = 17^0, 56'$, erit $v = 100725$; $u = 107360$; $☊LM = \sigma = 16^0, 21'$; $s = -116^0, 16'$; $\vartheta - \psi = 13^0, 26'$. Nunc pro $d\varphi$ inveniendo

$lv = 5{,}0031373$

$lcd\zeta\sqrt{cp} = 13{,}5436916$

$lvv = 10{,}0062746$

$l - d\varphi = 3{,}5374170$

$4{,}6855749$

$l - d\varphi = 8{,}2229919$

priori valore in mutatione angulorum, posteriori longitudinum est utendum.

Nunc pro angulis λ et μ inveniendis erit

$l\cos(\vartheta-\psi) = 9{,}9879525$

$l\cos\sigma = 9{,}9820721$

$l\sin\sigma = 9{,}4494849$

$9{,}9700246$

$9{,}4374374$

$l\sin(\vartheta-\psi) = 9{,}3660750$

$l\cos\omega = 9{,}9783702$

$9{,}3444452$

$l\sin\sigma = 9{,}4494849$

$l\cos\sigma = 9{,}9820721$

$8{,}7939301$

$9{,}3265173$

$$\begin{aligned} &+0,93331 \\ &+0,06222 \\ \cos\lambda &= 0,99553 \\ \lambda &= 5^0,\, 25' \end{aligned}$$

$$\begin{aligned} &+0,27380 \\ &-0,21209 \\ \sin\mu &= 0,06171 \\ \mu &= 3^0,\, 32', \end{aligned}$$

unde distantia $MN = w$ ita invenitur

$$\begin{aligned} lv &= 5,0031373 \\ l\sin\lambda &= 8,9749624 \\ l\cos\lambda &= 9,9980563 \\ lv\sin\lambda &= 3,9780997 \\ lv\cos\lambda &= 5,0011936 \\ l\sin\nu &= 9,9040529 \\ lw &= 4,0740468 \\ l\tfrac{c}{w} &= 0,9259532 \\ l\tfrac{c^3}{w^3} &= 2,7778596 \\ \tfrac{c^3}{w^3} &= 599,597 \end{aligned}$$

$$\begin{aligned} u &= 107360 \\ v\cos\lambda &= 100275 \\ u - v\cos\lambda &= 7085 \\ lv\sin\lambda &= 3,9780997 \\ l(u - v\cos\lambda) &= 3,8503399 \\ l\operatorname{tang}\nu &= 10,1277598 \\ \nu &= 53^0,\, 18' \\ w &= 11859 \\ lu &= 5,0308425 \\ l\tfrac{c}{u} &= 9,9691575 \\ l\tfrac{c^3}{u^3} &= 9,9074725 \\ \tfrac{c^3}{u^3} &= 0,8081. \end{aligned}$$

ergo $\frac{c^3}{w^3} - \frac{c^3}{u^3} = 598,789$ et $l\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right) = 2,7772738.$

Pro variatione parametri p:

$$\begin{aligned} l\tfrac{v^3}{c^3} &= 0,0094179 \\ lu &= 5,0308425 \\ l\sin\mu &= 8,7897867 \\ l\ldots &\;\; 2,7772738 \\ l - d\varphi &= 8,2229919 \\ &\;\; 4,8303128 \end{aligned}$$

erit ergo

$$dp = 2n\,.\,67657$$

seu $\quad dp = +135314n$

minor quam die praecedente.

Pro variatione semiaxis transversi r:

$$\begin{aligned} l\tfrac{qrr}{p} &= 3,2404707 \\ l\tfrac{v^3}{c^3} &= 0,0094119 \\ l\tfrac{c^3}{w^3} &= 2,7778596 \\ l - d\varphi &= 8,2229919 \\ l - \sin s &= 9,9526685 \\ &\;\; 4,2034026 \\ \text{pars I} &= -2n\,.\,15963 \\ \text{pars II} &= +2n\,.\,83696 \\ dr &= +2n\,.\,67733 \\ dr &= 135466n \end{aligned}$$

$$\begin{aligned} l\tfrac{q}{p} &= 3,2404707 \\ lv &= 5,0031373 \\ l\cos\lambda &= 9,4980563 \\ l - \sin s &= 9,9526685 \\ &\;\; 8,1943328 \\ -\tfrac{qv}{p}\cos\lambda\sin s &= +0,01564 \\ \sin\mu &= 0,06171 \\ \ldots\ldots &\;\; 0,07735 \\ l\ldots\ldots &\;\; 8,8884603 \\ lu &= 5,0308425 \\ l\tfrac{v}{c} &= 0,0031373 \\ l - d\varphi\,() &= 1,0002657 \\ &\;\; 4,9227058 \end{aligned}$$

Pro excentricitatis q variatione,

$$\begin{aligned} ldp &= 5,1313428 \\ lqr &= \underline{3,2278867} \\ & \quad \underline{1,9034541} \\ & -80,067 \\ & \underline{+79,681} \end{aligned}$$

$$\begin{aligned} lp &= 4,9874160 \\ ldr &= \underline{5,1318304} \\ & \quad 0,1192464 \\ lqrr &= \underline{8,2278867} \\ & \quad 1,9013597 \end{aligned}$$

$$2dq = -0,386n \quad \text{et} \quad dq = -0,193n.$$

Pro variatione anguli $☊ LB = \alpha$:

$$\begin{aligned} lp &= 4,9874160 \\ l\varphi &= \underline{5,0031373} \\ l(1+q\cos s) &= 9,9842787 \\ l\cos\lambda &= 9,9980563 \\ l\cos s &= \underline{-9,6459619} \\ & -9,6282969 \end{aligned}$$

$$\begin{aligned} 2+q\cos s &= 1,96445 \\ l(2+q\cos s) &= 0,2932409 \\ l\sin\mu &= 8,7897867 \\ l\sin s &= \underline{-0,9526685} \\ & -9,0356961 \end{aligned}$$

$$\text{pars postrema} = -0,42491 - 0,10857 = -0,53348$$

$$\begin{aligned} l\,\text{part. postr.} &= -9,7271181 \\ l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) &= 2,7772738 \\ lu &= \underline{5,0308425} \\ & -7,5352344 \\ lp &= \underline{4,9874160} \\ & -2,5478184 \end{aligned}$$

$$\begin{aligned} \text{pars posterior} &= +353,85 \\ \text{pars prior} &= \underline{-265,35} \\ \text{aggreg.} &= +88,50 \end{aligned}$$

$$\begin{aligned} l\frac{c^3}{w^3} &= 2,7778596 \\ l\cos s &= \underline{-9,6459619} \\ & \underline{-2,4238215} \\ l\,\text{aggr.} &= 1,9469433 \\ l\frac{v^3}{c^3} &= 0,0094119 \\ ld\varphi &= \underline{-3,5374170} \\ & -5,4937722 \\ lq &= \underline{8,2278867} \\ & -7,2658859. \end{aligned}$$

Ergo $$d\alpha = -18445310n \text{ min. sec.}$$

Pro variatione nodi et inclinationis:

$$\begin{aligned} l\frac{u}{p} &= 0,0434265 \\ l\frac{v^3}{c^3} &= 0,0094119 \\ l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) &= 2,7772738 \\ ld\varphi &= -3,5374170 \\ l\sin\sigma &= 9,4494849 \\ l\sin(\vartheta-\psi) &= \underline{9,3660750} \\ & -5,1830891 \end{aligned}$$

Ergo $$\begin{aligned} d\psi &= \underline{+152436n \text{ min. sec.}} \\ ld\psi &= +5,1830891 \\ l\sin\omega &= 9,4884240 \\ l\cos\sigma &= \underline{0,5325872} \\ & +6,2041003 \end{aligned}$$

ergo $$d\omega = +1599927n \text{ min. sec.}$$

Calculus pro intervallo a 27 ad 28 Aprilis.

Cum sit $p = 97144$, $q = 0{,}0169$; $r = 100000$, et $\omega = 17^0, 56'$, erit $\varphi = 100750$; $u = 108931$; $\sigma = 15^0, 23'$, $s = -117^0, 14'$; $\vartheta - \psi = 14^0, 21'$.

$$l\varphi = 5{,}0032451$$
$$lu = 5{,}0371515$$
$$lp = 4{,}9874160$$
$$l\frac{u}{p} = 0{,}0497355$$

$$lcd\zeta\sqrt{cp} = 13{,}5436916$$
$$l\varphi\varphi = 10{,}0064902$$
$$l - d\varphi = 3{,}5372014$$
$$4{,}6855749$$
$$l - d\varphi = 8{,}2227763$$

Hinc pro angulis λ et μ

$$l\cos(\vartheta - \psi) = 9{,}9862340;$$
$$l\cos\sigma = 9{,}9841548$$
$$l\sin\sigma = 9{,}4236974$$
$$9{,}9703888$$
$$9{,}4099314$$

$$l\sin(\vartheta - \psi) = 9{,}3941794$$
$$l\cos\omega = 9{,}9783702$$
$$9{,}3725496$$
$$l\sin\sigma = 9{,}4236974$$
$$l\cos\sigma = 9{,}9841548$$
$$8{,}7962470$$
$$9{,}3567044$$

$$+ 0{,}93409$$
$$+ 0{,}06255$$
$$\cos\lambda = 0{,}99664$$
$$\lambda = 4^0, 42'$$

$$+ 0{,}25700$$
$$- 0{,}22735$$
$$\sin\mu = 0{,}02965$$
$$l\sin\mu = 8{,}4720247,$$

unde colligitur distantia w

$$l\varphi = 5{,}0032451$$
$$l\sin\lambda = 8{,}9134881$$
$$l\cos\lambda = 9{,}9985372$$
$$l\varphi\sin\lambda = 3{,}9167332$$
$$l\varphi\cos\lambda = 5{,}0017823$$
$$l\sin\nu = 9{,}8425548$$
$$lw = 4{,}0741784$$
$$l\frac{c}{w} = 0{,}9258216$$
$$l\frac{c^3}{w^3} = 2{,}7774648$$
$$\frac{c^3}{w^3} = 599{,}05$$

$$u = 108931$$
$$\varphi\cos\lambda = 100411$$
$$u - \varphi\cos\lambda = 8520$$
$$l\varphi\sin\lambda = 3{,}9167332$$
$$l(u - \varphi\cos\lambda) = 3{,}9304396$$
$$l\tan\nu = 9{,}9862936$$
$$\nu = 44^0, 6'$$
$$l\frac{c}{u} = 9{,}9628485$$
$$l\frac{c^3}{u^3} = 9{,}8885455$$
$$\frac{c^3}{u^3} = 0{,}7706$$
$$\frac{c^3}{w^3} - \frac{c^3}{u^3} = 598{,}27$$
$$l\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right) = 2{,}7768972$$

Pro variatione parametri p:

$$\begin{aligned}
l\frac{v^3}{c^3} &= 0{,}0097353\\
lu &= 5{,}0371515\\
l\sin\mu &= 8{,}4720247\\
l\ldots &= 2{,}7768972\\
l - d\varphi &= -8{,}2227763\\
&\ +4{,}5185851
\end{aligned}$$

Ergo

$$\begin{aligned}
dp &= 2n\,.\,33005\\
\text{seu}\quad dp &= +66010n\\
ldp &= 4{,}8196097 + ln
\end{aligned}$$

Pro variatione semiaxis transversi r:

$$\begin{aligned}
l\frac{qrr}{p} &= 3{,}2404707\\
l(v^3:c^3) &= 0{,}0097353\\
l(c^3:w^3) &= 2{,}7774648\\
ld\varphi &= -8{,}2227763\\
l\sin s &= -9{,}9489752\\
&\ +4{,}1994223
\end{aligned}$$

$$\begin{aligned}
\text{pars I} &= -2n\,.\,15828\\
\text{pars II} &= +2n\,.\,49547\\
dr &= +2n\,.\,33719\\
dr &= +67438n
\end{aligned}$$

$$\begin{aligned}
l\frac{q}{p} &= 3{,}2404707\\
lv &= 5{,}0032451\\
-l\cos\lambda &= -9{,}9985372\\
l\sin s &= -9{,}9489752\\
&\ -8{,}1912282\\
&\ +0{,}01553\\
\sin\mu &= 0{,}02965\\
\ldots\ &\ \ 0{,}04518\\
l\ldots\ &\ \ 8{,}6549462\\
lu &= 5{,}0371515\\
l\frac{v}{c} &= 0{,}0032451\\
ld\varphi(\ldots) &= -0{,}9996735\\
&\ -4{,}6950163
\end{aligned}$$

Pro variatione excentricitatis q:

$$\begin{aligned}
ldp &= 4{,}8196097\\
lqr &= 3{,}2278867\\
&\ \ 1{,}5917230\\
&\ -39{,}059n\\
&\ +38{,}764n\\
2dq &= -\ 0{,}295n \quad \text{et} \quad dq = -0{,}148n.
\end{aligned}$$

$$\begin{aligned}
lp &= 4{,}9874160\\
ldr &= 4{,}8289047\\
&\ \ 9{,}8163207\\
lqrr &= 8{,}2278867\\
&\ \ 1{,}5884340
\end{aligned}$$

Pro variatione anguli ☊$LB = \alpha$:

$$\begin{aligned}
lp &= -4{,}9874160\\
lv &= 5{,}0032451\\
l(1+q\cos s) &= 9{,}9841709\\
l\cos\lambda &= 9{,}9985372\\
l\cos s &= -9{,}6605005\\
&\ -9{,}6432086
\end{aligned}$$

$$\begin{aligned}
2+q\cos s &= 1{,}96421\\
l(2+q\cos s) &= 0{,}2931857\\
l\sin\mu &= 8{,}4720247\\
l\sin s &= -9{,}9489752\\
&\ -8{,}7141856
\end{aligned}$$

pars postrema $= - 0,43975 - 0,05178 = - 0,49153.$

l pars postr. $= - 9,6915500$

$l\left(\frac{c^3}{v^3} - \frac{c^3}{u^3}\right) = 2,7768972$

$l\frac{p}{u} = - 0,0497355$

$2,5181827$

pars posterior $= + 329,748$

pars prior $= - 274,136$

aggreg. $= + 55,612$

$d\alpha = - 11593600n$ min. sec.

$l\frac{c^3}{v^3} = 2,7774648$

$l\cos s = - 9,6605005$

$- 2,4379653$

l aggr. $= 1,7451685$

$l\frac{v^3}{c^3} = 0,0097353$

$l - d\varphi = - 3,5372014$

$l\frac{1}{q} = 1,7721133$

$- 7,0642185$

Pro variatione nodi et inclinationis:

$l\frac{u}{p} = 0,0497355$

$l\frac{v^3}{c^3} = 0,0097353$

$l\left(\frac{c^3}{v^3} - \frac{c^3}{u^3}\right) = 2,7768972$

$l d\varphi = - 3,5372014$

$l\sin\sigma = 9,4236974$

$l\sin(\vartheta - \psi) = 9,3941794$

$- 5,1914462$

Ergo $d\psi = + 155398n$ min. sec.

$l d\psi = 5,1914462$

$l\sin\omega = 9,4884240$

$4,6798702$

$l\,\mathrm{tang}\,\varpi = 9,4395426$

$5,2403276$

ergo $d\omega = 173911n$ min. sec.

Calculus pro intervallo a 28 ad 29 Aprilis.

Hic erit $v = 100775$; $u = 110550$; $\sigma = 14^0\,25'$; $s = -118^0\,12'$ et $\vartheta - \psi = 15^0 14'$, unde pro $d\varphi$ inveniendo

$lv = 5,0033528$

$lu = 5,0435587$

$lp = 4,9874160$

$l\frac{u}{p} = 0,0561427$

$l\,cd\zeta\sqrt{cp} = 13,5436916$

$l\varphi = 10,0067056$

$l d\varphi = - 3,5369860$

$4,6855709$

$l d\varphi = - 8,2225609$

nunc pro angulis λ et μ

$l\cos(\vartheta - \psi) = 9,9844660$

$l\cos\sigma = 9,9861045$

$l\sin\sigma = 9,3961499$

$9,9705705$

$9,3806159$

$+ 0,93448$

$+ 0,06224$

$\cos\lambda = 0,99672$

$\lambda = 4^0\,38'$

$l\sin(\vartheta - \psi) = 9,4195436$

$l\cos\omega = 9,9783702$

$9,3979138$

$l\sin\sigma = 9,3961499$

$l\cos\sigma = 9,9861045$

$8,7940637$

$9,3840183$

$+ 0,24022$

$- 0,24211$

$\sin\mu = - 0,00189$

$l\sin\mu = - 7,2764618$

unde colligitur distantia $MN = \varpi$ hoc modo

$$l\nu = 5{,}0033528$$
$$l\sin\lambda = 8{,}9072975$$
$$l\cos\lambda = 9{,}9985784$$
$$l\nu\sin\lambda = 3{,}9106503$$
$$l\nu\cos\lambda = 5{,}0019312$$
$$l\sin\nu = 9{,}7974640$$
$$l\varpi = 4{,}1131863$$
$$l\frac{c}{w} = 0{,}8868136$$
$$l\frac{c^3}{w^3} = 2{,}6604408$$
$$\frac{c^3}{w^3} = 457{,}55$$

$$u = 110550$$
$$\nu\cos\lambda = 100446$$
$$\ldots\ldots = 10104$$
$$l\nu\sin\lambda = 3{,}9106503$$
$$l\ldots\ldots = 4{,}0044933$$
$$l\,\mathrm{tang}\,\nu = 9{,}9061570$$
$$\nu = 38^0\,51'$$
$$l\frac{c}{u} = 9{,}9564413$$
$$l\frac{c^3}{u^3} = 9{,}8693239$$
$$\frac{c^3}{u^3} = 0{,}740.$$
$$\frac{c^3}{w^3} - \frac{c^3}{u^3} = 456{,}81$$
$$l\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right) = 2{,}6597356$$

Pro variatione parametri p:

$$l\frac{\nu^3}{c^3} = 0{,}0100584$$
$$lu = 5{,}0435587$$
$$l\sin\mu = -7{,}2764618$$
$$l\ldots\ldots = 2{,}6597356$$
$$ld\varphi = -9{,}2225609$$
$$+3{,}2123754$$

Ergo $dp = -2n \cdot 1631$

seu $dp = -3262n$

$$ldp = -3{,}5134054$$

Pro variatione semiaxis transversi r:

$$l\frac{qrr}{p} = 3{,}2404707$$
$$l\frac{\nu^3}{c^3} = 0{,}0100584$$
$$l\frac{c^3}{w^3} = 2{,}6604408$$
$$ld\varphi = -8{,}2225609$$
$$l\sin s = -9{,}9451255$$
$$+4{,}0786563$$
$$\text{pars I} = -2n \,.\, 11985$$
$$\text{pars II} = +2n \,.\, 11478$$
$$dr = -2n \,.\, 507$$

seu $dr = -1014n$

$$l\frac{q}{p} = 3{,}2404707$$
$$l\nu\cos\lambda = 5{,}0019312$$
$$l\sin s = -9{,}9451255$$
$$-8{,}1875274$$
$$\ldots\ldots\quad +0{,}01540$$
$$\sin\mu = -0{,}00189$$
$$\ldots\ldots\quad 0{,}01351$$
$$l\ldots\ldots\quad 8{,}1306553$$
$$lu = 5{,}0435587$$
$$l\frac{\nu}{c} = 0{,}0033528$$
$$ld\varphi = -8{,}2225609$$
$$l\left(\frac{c^3}{\nu^3} - \frac{c^3}{u^3}\right) = 2{,}6597356$$
$$-4{,}0598633$$
$$lp = 4{,}9874160$$
$$ldr = 3{,}0060380$$
$$-7{,}9934540$$
$$lqrr = 8{,}2278867$$
$$-9{,}7655673$$

$$ldp = -3{,}5134054$$
$$lqr = 3{,}2278867$$
$$+0{,}2855187$$
$$+1{,}9298$$
$$-0{,}5829$$

$$2\,dq = +1{,}3469n \quad \text{et} \quad dq = +0{,}6735n.$$

Pro variatione anguli $\Omega LB = \alpha$:

$$\begin{array}{rl} lp = & 4,9874160 \\ l\varrho = & 5,0033528 \\ \hline l(1+q\cos s) = & 9,9840632 \\ l\cos\lambda = & 9,9985784 \\ l\cos s = & -9,6744485 \\ \hline & -9,6570901 \end{array} \qquad \begin{array}{rl} 2+q\cos s = & 1,96397 \\ \hline l(2+q\cos s) = & 0,2931349 \\ l\sin\mu = & -7,2764618 \\ l\sin s = & -9,9451255 \\ \hline & +7,5147222 \end{array}$$

$$\text{pars postrema} = -0,45404 + 0,00327 = -0,45077$$

$$\begin{array}{rl} l\text{ partis postr.} = & -9,6539550 \\ l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) = & +2,6597356 \\ l\frac{u}{p} = & 0,0561427 \\ \hline & -2,3698333 \\ & \\ \text{pars post.} = & +234,33 \\ \text{pars prior} = & -216,21 \\ \hline \text{aggreg.} = & +18,12 \end{array} \qquad \begin{array}{rl} l\frac{c^3}{w^3} = & 2,6604408 \\ l\cos s = & -9,6744485 \\ \hline & -2,3348893 \\ \hline l\text{ aggr.} = & 1,2581582 \\ l\frac{\varrho^3}{c^3} = & 0,0100584 \\ ld\varphi = & -3,5369860 \\ l\frac{1}{q} = & 1,7721133 \\ \hline & -6,5773159. \end{array}$$

Ergo $$d\alpha = -3778470\,n \text{ min. sec.}$$

Pro variatione nodi et inclinationis:

$$\begin{array}{rl} l\frac{u}{p} = & 0,0561427 \\ l\frac{\varrho^3}{c^3} = & 0,0100584 \\ l\ldots = & 2,6597356 \\ ld\varphi = & -3,5369860 \\ l\sin\sigma\sin(\vartheta-\psi) = & 8,8156935 \\ \hline l-d\psi = & -5,0786162 \end{array} \qquad \begin{array}{lrl} \text{Ergo} & d\psi = & +119840\,n \text{ min. sec.} \\ \hline & ld\psi = & +5,0786162 \\ & l\sin\omega = & 9,4884240 \\ & l\cot\sigma = & 0,5899546 \\ \hline & & +5,1569948 \\ \text{ergo} & d\omega = & 143550\,n \text{ min. sec.} \end{array}$$

Calculus pro intervallo a 29 ad 30 Aprilis.

Hic erit $\varrho = 100800$; $u = 112155$; $\sigma = 13^0\,23'$, $s = -119^0\,10'$ et $\vartheta - \psi = 16^0\,6'$, unde pro $d\varphi$ inveniendo

$$\begin{array}{rl} l\varrho = & 5,0034605 \\ lu = & 5,0498200 \\ lp = & 4,9874160 \\ \hline l\frac{u}{p} = & 0,0624040 \end{array} \qquad \begin{array}{rl} lcd\zeta\sqrt{cp} = & 13,5436916 \\ l\varrho\varrho = & 10,0069210 \\ \hline ld\varphi = & -3,5367706 \\ & 4,6855749 \\ \hline ld\varphi = & -8,2223455 \end{array}$$

*

nunc pro angulis λ et μ

$$\begin{aligned}
l\cos(\vartheta-\psi) &= 9{,}9826236\\
l\cos\sigma &= 9{,}9879223\\
l\sin\sigma &= 9{,}3666036\\
&\ 9{,}9705459\\
&\ 9{,}3492272\\
&\ +0{,}93443\\
&\ +0{,}06137\\
\cos\lambda &= 0{,}99580\\
\lambda &= 5^0\,15'
\end{aligned}$$

$$\begin{aligned}
l\sin(\vartheta-\psi) &= 9{,}4429728\\
l\cos\omega &= 9{,}9783702\\
&\ 9{,}4213430\\
l\sin\sigma &= 9{,}3666036\\
l\cos\sigma &= 9{,}9879223\\
&\ 8{,}7879466\\
&\ 9{,}4092653\\
&\ +0{,}22347\\
&\ -0{,}25660\\
\sin\mu &= -0{,}03313\\
l\sin\mu &= -8{,}5203525,
\end{aligned}$$

unde colligitur distantia $MN = \varpi$

$$\begin{aligned}
l\nu &= 5{,}0034605\\
l\sin\lambda &= 8{,}9614288\\
l\cos\lambda &= 9{,}9981743\\
l\nu\sin\lambda &= 3{,}9648893\\
l\nu\cos\lambda &= 5{,}0016348\\
l\sin\nu &= 9{,}7899880\\
l\varpi &= 4{,}1749013\\
l\frac{c}{w} &= 0{,}8250987\\
l\frac{c^3}{w^3} &= 2{,}4752961\\
\frac{c^3}{w^3} &= 298{,}74\\
\frac{c^3}{w^3}-\frac{c^3}{u^3} &= 298{,}03
\end{aligned}$$

$$\begin{aligned}
u &= 112155\\
\nu\cos\lambda &= 100377\\
\dots\dots &\ 11778\\
l\nu\sin\lambda &= 3{,}9648893\\
l\,\dots\dots &\ 4{,}0710715\\
l\,\mathrm{tang}\,\nu &= 9{,}8938178\\
\nu &= 38^0\,4'\\
l\frac{c}{u} &= 9{,}9501800\\
l\frac{c^3}{u^3} &= 9{,}8505400\\
\frac{c^3}{u^3} &= 0{,}7088\\
l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) &= 2{,}4742600.
\end{aligned}$$

Pro variatione semiparametri p:

$$\begin{aligned}
l\frac{\nu^3}{c^3} &= 0{,}0103815\\
l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) &= 2{,}4742600\\
lu &= 5{,}0498200\\
l\sin\mu &= -8{,}5203525\\
ld\varphi &= -8{,}2223455\\
&\ +4{,}2771595
\end{aligned}$$

Ergo $dp = -2n\,.\,18930$

seu $dp = -37860\,n$

et $ldp = -4{,}5781895$

Pro variatione semiaxis transversi r:

$$
\begin{aligned}
l\frac{q}{p} &= 3,2404707 \\
l\frac{\rho^3}{c^3} &= 0,0103815 \\
l\frac{c^3}{w^3} &= 2,4752961 \\
l\,d\varphi &= -\,8,2223455 \\
l\sin s &= -\,9,9411166 \\
\hline
& +\,3,8896104
\end{aligned}
$$

$$
\begin{aligned}
\text{pars I} &= -\,2n.7756 \\
\text{pars II} &= -\,2n.10052 \\
\hline
dr &= -\,2n.17808 \\
dr &= -\,35616n
\end{aligned}
$$

$$
\begin{aligned}
l\frac{q}{p} &= 3,2404707 \\
l\rho\cos\lambda &= 5,0016348 \\
l\sin s &= -\,9,9411166 \\
\hline
& -\,8,1832221 \\
\hline
l\cos s &= -\,9,6878425 \\
\frac{-\,q\rho\cos\lambda\sin s}{p} &= +\,0,01525 \\
\sin\mu &= -\,0,03313 \\
\hline
\text{aggreg.} & -\,0,01788 \\
\hline
l\,\text{aggreg.} &= -\,8,2523675 \\
l\frac{u}{c} &= 5,0532805 \\
l-d\varphi &= 8,2223455 \\
l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) &= 2,4742600 \\
\hline
& -\,4,0022535.
\end{aligned}
$$

Cum ex p et r datur $q=\sqrt{\left(1-\frac{p}{r}\right)}$, non opus est quaerere dq.

Pro variatione anguli $\Omega LB=\alpha$:

$$
\begin{aligned}
lp &= 4,9874160 \\
l\rho &= 5,0034605 \\
\hline
l(1+q\cos s) &= 9,9839555 \\
l\cos\lambda &= 9,9981743 \\
l\cos s &= -\,9,6878425 \\
\hline
& -\,9,6699723
\end{aligned}
$$

$$
\begin{aligned}
2+q\cos s &= 1,96373 \\
\hline
l(2+q\cos s) &= 0,2930751 \\
l\sin\mu &= -\,8,5203525 \\
l\sin s &= -\,9,9411166 \\
\hline
& +\,8,7545442
\end{aligned}
$$

$$
\text{pars postrema} = \begin{matrix} -\,0,46771 \\ +\,0,05683 \end{matrix} = -\,0,41088
$$

$$
\begin{aligned}
l\,\text{pars postr.} &= -\,9,6137150 \\
l\frac{u}{p} &= -\,0,0624040 \\
l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) &= 2,4742600 \\
\hline
& +\,2,1503790
\end{aligned}
$$

$$
\begin{aligned}
\text{pars posterior} &= +\,141,38 \\
\text{pars prior} &= -\,145,59 \\
\hline
\text{aggreg.} &= -\quad 4,21
\end{aligned}
$$

$$
\begin{aligned}
l\frac{c^3}{w^3} &= 2,4752961 \\
l\cos s &= -\,9,6878425 \\
\hline
& -\,2,1631386 \\
\hline
l\,\text{aggr.} &= -\,0,6242821 \\
l\frac{\rho^3}{c^3} &= 0,0103815 \\
l\,d\varphi &= -\,3,5367706 \\
l\frac{1}{q} &= 1,7721133 \\
\hline
& +\,5,9435475
\end{aligned}
$$

$$
d\alpha = +\,878107n \text{ min. sec.}
$$

Pro variatione nodi et inclinationis:

$$l\frac{u}{p} = 0{,}0624040$$
$$l\frac{\nu^3}{c^3}\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) = 2{,}4846415$$
$$ld\varphi = -3{,}5367706$$
$$l\sin\sigma\sin(\vartheta-\psi) = 8{,}8095764$$
$$-4{,}8933925$$

ergo $d\psi = +78234n$ min. sec.

$$ld\psi = 4{,}8933925$$
$$l\sin\omega = 9{,}4884240$$
$$l\cot\sigma = 0{,}6213187$$
$$ld\omega = 5{,}0031352$$

ergo $d\omega = +100724n$ min. sec.

Calculus pro intervallo a 30 Aprilis ad 1 Maji.

Hic erit $\nu = 100825$; $u = 113745$; $\sigma = 12^0\,29'$; $s = -120^0\,8'$ et $\vartheta - \psi = 16^0\,56'$, unde pro $d\varphi$ inveniendo

$$l\nu = 5{,}0035682$$
$$lu = 5{,}0559323$$
$$lp = 4{,}9874160$$
$$l\frac{u}{p} = 0{,}0685163$$

$$lcd\zeta\sqrt{cp} = 13{,}5436916$$
$$l\nu\nu = 10{,}0071364$$
$$ld\varphi = -3{,}5365552$$
$$4{,}6855749$$
$$ld\varphi = -8{,}2221301.$$

Nunc pro angulis λ et μ

$$l\cos(\vartheta-\psi) = 9{,}9807505$$
$$l\cos\sigma = 9{,}9896095$$
$$l\sin\sigma = 9{,}3347665$$
$$9{,}9703600$$
$$9{,}3155170$$
$$+0{,}93403$$
$$+0{,}05990$$
$$\cos\lambda = +0{,}99393$$
$$\lambda = 6^0\,38'$$

$$l\sin(\vartheta-\psi) = 9{,}4642790$$
$$l\cos\omega = 9{,}9783702$$
$$9{,}4426492$$
$$l\sin\sigma = 9{,}3347665$$
$$l\cos\sigma = 9{,}9896095$$
$$8{,}7774157$$
$$9{,}4322587$$
$$+0{,}20678$$
$$-0{,}27055$$
$$\sin\mu = -0{,}06377$$
$$l\sin\mu = -8{,}8046164,$$

unde colligitur distantia $MN = w$

$$l\nu = 5{,}0035682$$
$$l\cos\lambda = 9{,}9973554$$
$$l\sin\lambda = 9{,}0414852$$
$$l\nu\cos\lambda = 5{,}0009236$$
$$l\nu\sin\lambda = 4{,}0450534$$
$$l\sin\nu = 9{,}8019735$$
$$lw = 4{,}2430799$$
$$l\frac{c}{w} = 0{,}7569201$$
$$l\frac{c^3}{w^3} = 2{,}2707603$$
$$\frac{c^3}{w^3} = 186{,}536$$

$$u = 113745$$
$$\nu\cos\lambda = 100213$$
$$u-\nu\cos\lambda = 13532$$
$$l\nu\sin\lambda = 4{,}0450534$$
$$l(u-\nu\cos\lambda) = 4{,}1313620$$
$$l\,\mathrm{tang}\,\nu = 9{,}9136914$$
$$\nu = 39^0\,20'$$
$$l\frac{c}{u} = 9{,}9440677$$
$$l\frac{c^3}{u^3} = 9{,}8322031$$
$$\frac{c^3}{u^3} = 0{,}680.$$

ergo $\frac{c^3}{w^3}-\frac{c^3}{u^3} = 185{,}856$ et $l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) = 2{,}2691700.$

ro variatione semiparametri p:

$$l\frac{v^3}{c^3} = 0{,}0107046$$
$$l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) = 2{,}2691700$$
$$lu = 5{,}0559323$$
$$l\sin\mu = -8{,}8046164$$
$$ld\varphi = -8{,}2221301$$
$$+4{,}3625534$$

Ergo
$$dp = -2n\,.\,23044$$
seu $$dp = -46088n$$

ro variatione semiaxis transversi r:

$$l\frac{qrr}{p} = 3{,}2404707$$
$$l\frac{v^3}{c^3} = 0{,}0107046$$
$$l\frac{c^3}{w^3} = 2{,}2707603$$
$$ld\varphi = -8{,}2221301$$
$$l\sin s = -9{,}9369456$$
$$+3{,}6810113$$

$$\text{pars I} = -2n\,.\;4797{,}5$$
$$\text{pars II} = -2n\,.\,17308$$
$$dr = -2n\,.\,22105$$
$$dr = -44210n$$

$$l\frac{q}{p} = 3{,}2404707$$
$$lv\cos\lambda = 0{,}0009236$$
$$l\sin s = -9{,}9369456$$
$$-8{,}1783399$$
$$+0{,}01508$$
$$\sin\mu = -0{,}06377$$
$$\ldots\ldots = -0{,}04869$$
$$l\ldots\ldots = -8{,}6874398$$
$$lu = 5{,}0559323$$
$$l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) = 2{,}2691700$$
$$ld\varphi = -8{,}2221301$$
$$l\frac{v}{c} = 0{,}0035682$$
$$+4{,}2382404$$

ro variatione anguli ☊$LB = \alpha$:

$$lp = 4{,}9874160$$
$$lv = 5{,}0035682$$
$$l(1+q\cos s) = 9{,}9838478$$
$$l\cos\lambda = 9{,}9973554$$
$$l\cos s = -9{,}7007158$$
$$-9{,}6819190$$

$$2+q\cos s = 1{,}9635$$
$$l(2+q\cos s) = 0{,}2930309$$
$$l\sin\mu = -8{,}8046164$$
$$l\sin s = -9{,}9369456$$
$$+9{,}0345929$$

$$\text{pars postrema} = \frac{-0{,}48075}{+0{,}10829} = -0{,}37246.$$

$$l\,\text{part. postr.} = -9{,}5710796$$
$$l\frac{u}{p} = 0{,}0685163$$
$$l\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right) = 2{,}2691700$$
$$l\,\text{part. post.} = -1{,}9087659$$
$$\text{pars posterior} = +81{,}052$$
$$\text{pars prior} = -89{,}676$$
$$\text{aggreg.} = -\;8{,}624$$

$$l\frac{c^3}{w^3} = 2{,}2707603$$
$$l\cos s = -9{,}6819190$$
$$l\,\text{part. I} = -1{,}9526793$$
$$l\,\text{aggreg.} = -0{,}9357087$$
$$l\frac{v^3}{c^3} = 0{,}0107046$$
$$ld\varphi = -3{,}5365552$$
$$l\frac{1}{q} = 1{,}7721133$$
$$+6{,}2550818$$

$$d\alpha = +1799210n \text{ min. sec.}$$

Pro variatione nodi et inclinationis

$$l\frac{u}{p} = 0,0685163$$
$$l\rho^3\left(\frac{1}{w^3}-\frac{1}{u^3}\right) = 2,2798746$$
$$ld\varphi = -3,5365552$$
$$l\sin\sigma\sin(\vartheta-\psi) = 8,7990455$$
$$l-d\psi = -4,6839916$$

Ergo
$$d\psi = +48305n \text{ min sec}$$
$$ld\psi = 4,6839916$$
$$l\sin\omega = 9,4884240$$
$$l\cos\sigma = 0,6548430$$
$$ld\omega = 4,8272586$$
$$d\omega = 67183n \text{ min. sec}$$

Calculus pro intervallo a 1 ad 2 Maji.

Hic erit $\rho = 100850$; $u = 115380$; $\sigma = 11^0\,31'$, $s = -121^0\,6'$, et $\vartheta - \psi = 17^0\,45'$, unde pro $d\varphi$ inveniendo

$$l\rho = 5,0036759$$
$$lu = 5,0621305$$
$$lp = 4,9874160$$
$$l\frac{u}{p} = 0,0747145$$

$$lcd\zeta\sqrt{cp} = 13,5436916$$
$$l\rho\rho = 10,0073518$$
$$ld\varphi = -3,5363398$$
$$4,6855749$$
$$ld\varphi = -8,2219147.$$

Nunc pro angulis λ et μ

$$l\cos(\vartheta-\psi) = 9,9788175$$
$$l\cos\sigma = 9,9911670$$
$$l\sin\sigma = 9,3002758$$
$$9,9699845$$
$$9,2790933$$

$$+0,93322$$
$$+0,05791$$
$$\cos\lambda = 0,99113$$
$$\lambda = 7^0\,38'$$

$$l\sin(\vartheta-\psi) = 9,4841066$$
$$l\cos\omega = 9,9783702$$
$$9,4624768$$
$$l\sin\sigma = 9,3002758$$
$$l\cos\sigma = 9,9911670$$
$$8,7627526$$
$$9,4536438$$

$$+0,19015$$
$$-0,28421$$
$$\sin\mu = -0,09406$$
$$l\sin\mu = -8,9734050.$$

unde colligitur distantia $MN = w$:

$$l\rho = 5,0036759$$
$$l\cos\lambda = 9,9961343$$
$$l\sin\lambda = 9,1233061$$
$$l\rho\cos\lambda = 4,9998102$$
$$l\rho\sin\lambda = 4,1269820$$
$$l\sin\nu = 9,8166521$$
$$lw = 4,3103299$$
$$l\frac{c}{w} = 0,6896701$$
$$l\frac{c^3}{w^3} = 2,0690103$$
$$\frac{c^3}{w^3} = 117,22$$
$$\frac{c^3}{w^3} - \frac{c^3}{u^3} = 116,75$$

$$u = 115380$$
$$\rho\cos\lambda = 99956$$
$$u - \rho\cos\lambda = 15424$$
$$l\rho\sin\lambda = 4,1269820$$
$$l(u-\rho\cos\lambda) = 4,1881970$$
$$l\tang\nu = 9,9387850$$
$$\nu = 40^0\,58'$$
$$l\frac{c}{u} = 9,9378695$$
$$l\frac{c^3}{u^3} = 9,8136085$$
$$\frac{c^3}{u^3} = 0,65$$

et $l\left(\frac{c^3}{w^3} - \frac{c^3}{u^3}\right) = 2,0665868.$

Pro variatione semiparametri p:

$l\frac{v^3}{c^3} = \quad 0{,}0110277$

$l \ldots = \quad 2{,}0665868$

$lu = \quad 5{,}0621305$

$l \sin \mu = -\, 8{,}9734050$

$l d\varphi = -\, 8{,}2219147$

$+\, 4{,}3350647$

Ergo

$dp = -\, 2n \,.\, 21630$

seu $\quad dp = -\, 43260\, n$

Pro variatione semiaxis r:

$l\frac{qrr}{p} = \quad 3{,}2404707$

$l\frac{v^3}{c^3} = \quad 0{,}0110277$

$l\frac{c^3}{w^3} = \quad 2{,}0690103$

$l d\varphi = -\, 8{,}2219147$

$l \sin s = -\, 9{,}9326092$

$+\, 3{,}4750326$

pars I $= -\, 2n \,.\; 2985{,}6$

pars II $= -\, 2n \,.\, 17900$

$dr = -\, 2n \,.\, 20886$

$dr = -\, 41772\, n$

$l\frac{q}{p} = \quad 3{,}2404707$

$l v \cos \lambda = \quad 4{,}9998102$

$l \sin s = -\, 9{,}9326092$

$-\, 8{,}1728901$

$+\, 0{,}01489$

$\sin \mu = -\, 0{,}09406$

$-\, 0{,}07917$

$l \ldots = -\, 8{,}8985606$

$l\frac{uv}{c} = \quad 5{,}0658064$

$l d\varphi = -\, 8{,}2219147$

$l \ldots = \quad 2{,}0665868$

$+\, 4{,}2528685$

Pro variatione anguli $\Omega LB = \alpha$:

$lp = \quad 4{,}9874160$

$lv = \quad 5{,}0036759$

$l(1 + q \cos s) = \quad 9{,}9837401$

$l \cos \lambda = \quad 9{,}9961343$

$l \cos s = -\, 9{,}7130983$

$-\, 9{,}6929727$

$2 + q \cos s = \quad 1{,}9633$

$l(2 + q \cos s) = \quad 0{,}2929867$

$l \sin \mu = -\, 8{,}9734050$

$l \sin s = -\, 9{,}9326092$

$+\, 9{,}1990009$

$$\text{pars postrema} = \frac{-\,0{,}49314}{+\,0{,}15812} = -\,0{,}33502.$$

l part. postr. $= -\, 9{,}5250707$

$l \ldots = \quad 2{,}0665868$

$l\frac{u}{p} = \quad 0{,}0747145$

$-\, 1{,}6662720$

pars posterior $= +\, 46{,}373$

pars prior $= -\, 60{,}549$

aggreg. $= -\, 14{,}176$

$l\frac{c^3}{w^3} = \quad 2{,}0690103$

$l \cos s = -\, 9{,}7130983$

$-\, 1{,}7821086$

l aggreg. $= -\, 1{,}1515537$

$l\frac{v^3}{c^3} = \quad 0{,}0110277$

$l d\varphi = -\, 3{,}5365398$

$l\frac{1}{q} = \quad 1{,}7721133$

$+\, 6{,}4712345$

Ergo $\quad d\alpha = +\, 2959610\, n$ min. sec.

Pro variatione nodi et inclinationis:

$$l\frac{u}{p}\left(\frac{c^3}{w^3}-\frac{c^3}{u^3}\right)= \quad 2,1413013$$

$$l\frac{\nu^3}{c^3}= \quad 0,0110277$$

$$ld\varphi=-3,5365398$$

$$l\sin\sigma\sin(\vartheta-\psi)= \quad 8,7843824$$

$$-4,4732512$$

ergo $d\psi=+29734n$ min. sec.

$$ld\psi= \quad 4,4732512$$

$$l\sin\omega= \quad 9,4889240$$

$$l\cot\sigma= \quad 0,6908912$$

$$4,6525664$$

ergo $d\omega= \quad 44933n$ min. sec.

Conclusio.

Cum variationes inventae sint admodum notabiles, simili modo tam ante terminum 25 Aprilis quam post 2 Maj. definiri debent. Quas igitur computavi hic simul aspectui exponam:

Intervallum Aprilis	dp	dr	da	$d\psi$	$d\omega$
15 — 16					
16 — 17					
17 — 18					
18 — 19					
19 — 20					
20 — 21	$+45310n$	$+144570n$	$-4937360n$	$+8903n$	$+6717n$
21 — 22					
22 — 23					
23 — 24					
24 — 25					
25 — 26	$+155650n$	$+154346n$	$-19059570n$	$+112880n$	$+111364n$
26 — 27	$+135314n$	$+135466n$	$-18445310n$	$+152436n$	$+159996n$
27 — 28	$+66010n$	$+67438n$	$-11593600n$	$+155398n$	$+173911n$
28 — 29	$-3262n$	$-1014n$	$-3778470n$	$+119840n$	$+143550n$
29 — 30	$-37860n$	$-35616n$	$+878107n$	$+78234n$	$+100724n$
30 — 1 Maj.	$-46088n$	$-44210n$	$+1799210n$	$+48305n$	$+67183n$
1 — 2	$-43260n$	$-41772n$	$+2959610n$	$+29734n$	$+44933n$
2 — 3					
3 — 4					
4 — 5					
5 — 6					
6 — 7	$-19031n$	$-17736n$	$+1554230n$	$+3608n$	$+9506n$

Si certiores essemus de elementis motus hujus cometae, operae pretium esset hunc calculum ulterius tam in antecedentia quam consequentia extendere; nunc autem sufficiat conjectura tantum perturbationes in motu terrae ortas crassa minerva colligere.

De variatione parametri.

Semiparameter p usque ad 28 Aprilis augetur, tum vero iterum minuitur; verumtamen augmenta multum praevalent. Videtur autem totum augmentum exsurgere ad $700000\,n$, unde cum ante cometae adventum fuerit semiparameter $p = 97144$, is deinceps erit $= 97144 + 700000\,n$. Quare si massa cometae aequalis esset massae terrae, ob $n = \frac{1}{200000}$, fieret is $= 97144 + 3\frac{1}{2}$; at si massa cometae ad massam terrae rationem $= m : 1$ habere ponatur, in postremum erit semiparameter $= 97144 + \frac{7m}{2}$.

De variatione axis transversi.

Semiaxis transversus r, qui ante cometae adventum sumtus est $= 100000$, fere similes mutationes patitur, quae autem aliquantillum erunt minores, ita ut augmentum totum aestimari queat quasi $= 690000\,n$, et semiaxis post discessum cometae $= 100000 + 690000\,n$. Hinc posita ratione massae cometae ad massam terrae $= m : 1$, erit semiaxis in posterum $= 100000 + \frac{69m}{20}$.

De variatione excentricitatis.

Cum sit in genere excentricitas $q = \sqrt{\left(1 - \frac{p}{r}\right)}$, eaque ante cometae adventum fuerit $= 0{,}0169$,

erit ea deinceps $\quad = \sqrt{\left(1 - \frac{97144 - 3{,}5m}{100000 + 3{,}45m}\right)} = \sqrt{(0{,}01856 - 0{,}0000015m)}$,

ideoque fiet excentricitas $\quad = 0{,}0169 - 0{,}000044m$

hoc est aliquanto minor quam ante. Quare si massa cometae centies superaret massam terrae, ut esset $m = 100$, foret excentricitas $= 0{,}0125$, maximaque solis aequatio multo minor esset futura.

De variatione anni solaris.

Ob auctum axem transversum quantitas anni solaris augebitur in ratione

$$1 : \left(1 + \frac{69m}{2000000}\right)^{\frac{3}{2}} = 1 : 1 + \frac{207m}{4000000}.$$

Cum igitur ante adventum cometae annus fuerit $365^d\,5^h\,49' = 525949'$, annus in posterum augmentum capiet $= 27m$ min. primorum. Dum ergo cometa esset terrae aequalis, annus 27 min. primis produceretur, fieretque $= 365^d\,6^h\,16'$. Ac si cometa adeo centies terram superaret, anni quantitas augmentum caperet 45 horarum, qui effectus sane foret stupendus.

De variatione lineae absidum.

Usque ad diem 29 Aprilis linea absidum maxime promovetur, tum vero iterum repellitur; sed promotio plurimum praevalet atque ad minimum $100000000\,n$ aestimanda videtur. Hinc si ut hactenus massa cometae m vicibus major ponatur, quam massa terrae, ab actione cometae linea absidum orbitae terrae per spatium $500\,m$ min. sec. promovebitur. Ergo si cometa terrae esset aequalis, haec promotio esset $= 8'\,20''$, sin autem centies esset major, foret ea $13^0\,23'\,20''$.

*

De variatione lineae nodorum et inclinationis.

Linea nodorum seu intersectio L☊ ab actione cometae super ejus orbita ad minimum pro-vebitur per spatium $950000\,n$ min. sec. et inclinatio fere tantundem augebitur: unde utraque p[illegible] turbatio erit $47\frac{1}{2}\,m$ min. sec., quae eo minus est dubia, cum actio cometae perpetuo augmen[illegible] producat.

Fig. 191. Consideremus haec elementa in coelo, sitque ☊C via cometae, ☊♎♋ ecliptica [illegible] adventum cometae, erit angulus ♎☊$C = 17^\circ\, 56' =$ ☊, et arcus ☊♎ $= 51^\circ\, 16'$, per ♎ transibit aequator $Æ$♎Q faciens cum ecliptica angulum $Æ$♎♋ $= 23^\circ\, 28\frac{1}{2}'$. Post effectum [illegible] cometae sit circulus $\varepsilon o \lambda \omega$ ecliptica secans priorem in o, erit ☊$\omega = d\psi$ et $C\omega o =$ ☊ $+ d\omega$. Duca-tur arculus ωu ad ☊o normalis, erit ☊$u = d\psi \cos$ ☊ et $\omega u = d\psi \sin$ ☊; ponatur tant[illegible] ☊$o = z$, erit $\sin z : \sin (z - d\psi \cos$ ☊$) = \sin($☊ $+ d\omega) : \sin$ ☊, unde fit $\operatorname{tang} z = \frac{d\psi \sin ☊}{d\omega}$ [illegible] ergo $d\omega = d\psi$, erit $l \operatorname{tang} z = l \sin$ ☊ $= 9{,}4884240$, ac propterea $z =$ ☊$o = 17^\circ\, 7'$; tum vero [illegible] $o \sin z = d\psi \sin$ ☊ erit $o = \frac{d\psi \sin ☊}{\sin z} = 1{,}0512\, d\psi = 50\,m$ min. sec., ob $d\psi = d\omega = 47\frac{1}{2}\,m$. Cum ergo sit ♎$o = -34^\circ\, 9'$, ecliptica quasi gyratur circa punctum ♏ $4^\circ\, 9'$ per angulum $50\,m$ min. sec., [illegible] ut punctum solstitiale ♋ magis ab aequatore removeatur et obliquitas eclipticae augeatur. Ducta ♎μ ad $o\omega$ normali, erit ♎$\mu = -50\,m \sin 34^\circ\, 9'$,

hincque
$$♎\lambda = \frac{-50\,m \,.\, \sin 34^\circ\, 9'}{\sin 23^\circ\, 28\frac{1}{2}'}, \qquad \text{et} \qquad \mu\lambda = \frac{-50\,m \,.\, \sin 34^\circ\, 9'}{\operatorname{tang} 23^\circ\, 28\frac{1}{2}'}.$$

Unde si obliquitas eclipticae pristina vocetur $= \varepsilon$ et nova $= \varepsilon + d\varepsilon$, erit

$$\sin \varepsilon : \sin (\varepsilon + d\varepsilon) = \sin (-34^\circ\, 9' - \mu\lambda) : \sin - 34^\circ\, 9' \quad \text{seu} \quad d\varepsilon = 50\,m \cos 34^\circ\, 9' = 41\,m \text{ sec.}$$

et cum sit $\mu\lambda = -63\,m$ sec. puncta aequinoctialia super ecliptica per $63\,m$ sec. promota erunt cen-senda, super aequatore autem per spatium $70\,m$ sec. In latitudine igitur stellarum, quarum longi-tudo est ♌ $4^\circ\, 9'$, iste effectus maximé spectabitur, dum stellarum borealium latitudo minuetur, austra-lium vero augebitur particula $50\,m$ sec. In stellis vero sub longitudine ♒ $4^\circ\, 9'$ sitis contrarium eveniet.

Si massa cometae multum superet massam terrae, hae perturbationes ad enormem quantitatem exsurgere poterunt, ita ut effectum non solum in Astronomia, sed etiam in vita communi simus sensuri. Quin etiam, cum de elementis orbitae cometae non simus satis certi, error in eam partem incidere posset, ut omnes hae perturbationes multo adeo majores essent prodituri, quam hic invenimus. Omnino autem etiamsi ob errores has perturbationes minui oporteret, et massa cometae minor esset quam terrae, tamen ab hoc tempore novam quasi epocham constitui conveniet, pro qua novae tabulae solares ante omnia essent condendae, quod negotium nonnisi pluribus elapsis annis perfici poterit. Lunares autem tabulae multo majorem ac difficiliorem emendationem requisiturae videntur.

Fig. 172. p. 185.

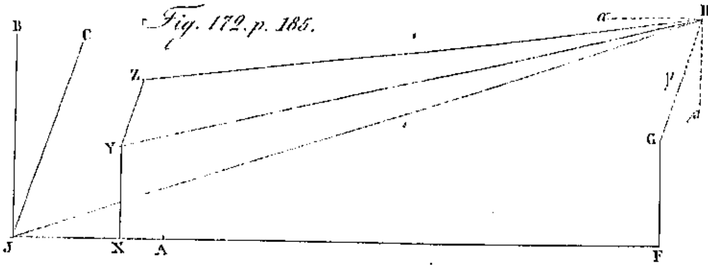


Fig. 173. p. 190.

Fig. 175. p. 194.

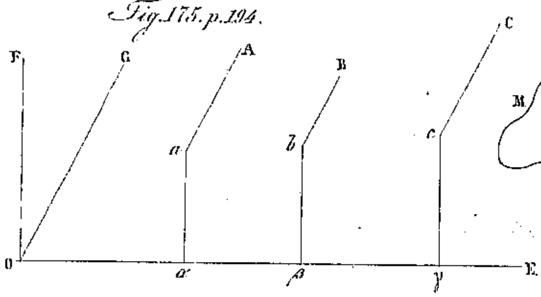


Fig. 174. p. 193.

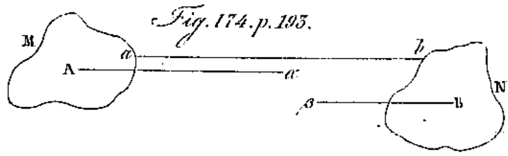


Fig. 176. p. 196.

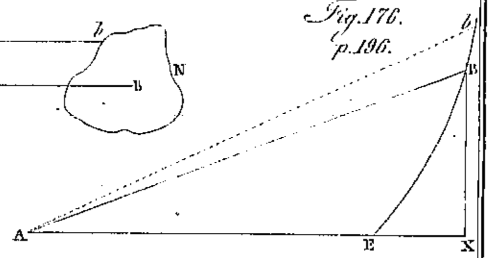


Fig. 179. p. 217.

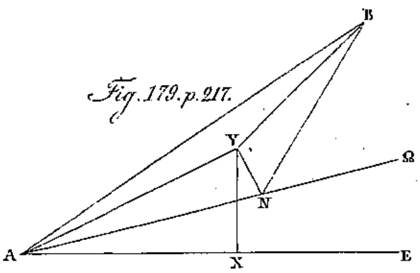


Fig. 177. p. 203.

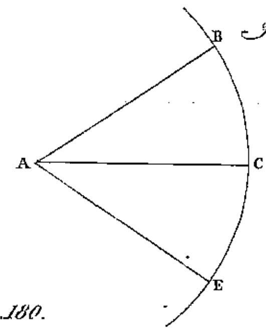


Fig. 178. p. 206.

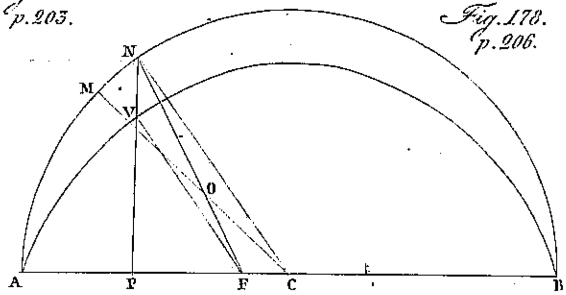


Fig. 180. p. 221.

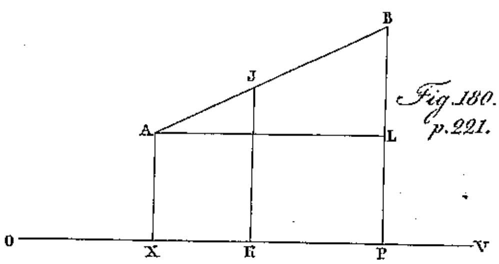


Fig. 182. p. 231.

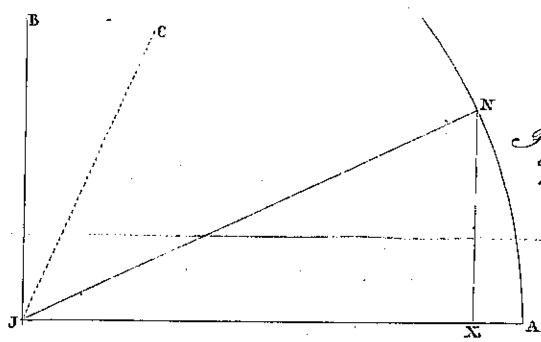


Fig. 184. p. 257.

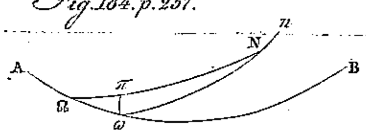


Fig. 181. p. 223.

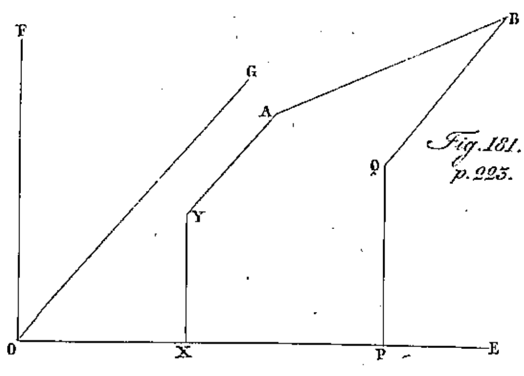
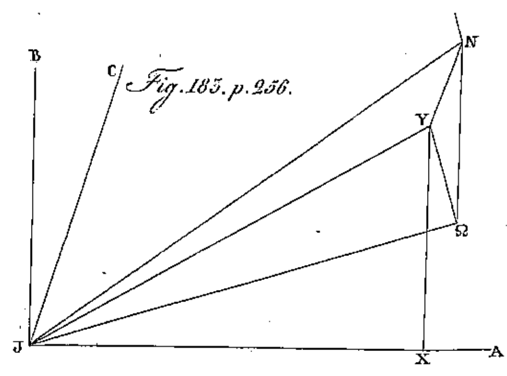


Fig. 183. p. 256.



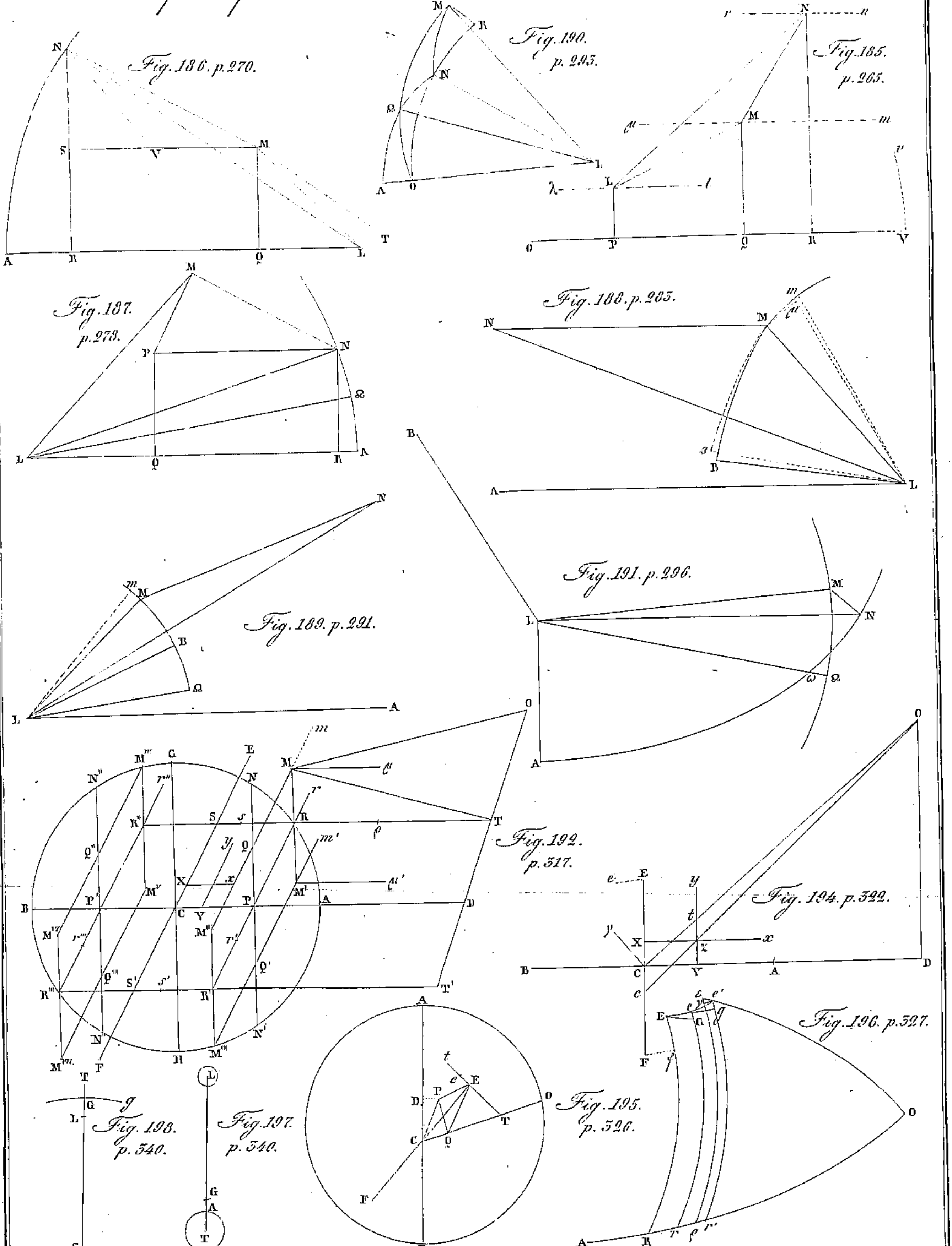


Fig. 185. p. 265.

Fig. 186. p. 270.

Fig. 187. p. 278.

Fig. 188. p. 283.

Fig. 189. p. 291.

Fig. 190. p. 293.

Fig. 191. p. 296.

Fig. 192. p. 317.

Fig. 194. p. 322.

Fig. 195. p. 326.

Fig. 196. p. 327.

Fig. 197. p. 340.

Fig. 198. p. 340.